(III) Abelian and non-abelian descent

X proj., smooth variety over a number field k. $X(\mathbb{A}_k) := \prod_{v \in \Omega_k} X(k_v)$

Let $B \subset \mathrm{Br}\, X$, define

$$X(\mathbb{A}_k)^B = \left\{ (P_v) \in X(\mathbb{A}_k),\ \forall \alpha \in B \ \sum_{v \in \Omega_k} j_v(\alpha(P_v)) = 0 \right\}$$

$j_v : \mathrm{Br}\, k_v \hookrightarrow \mathbb{Q}/\mathbb{Z}$

$(P_v) \in X(\mathbb{A}_k)^B$ = there is no Brauer-Manin obstruction associated to elements of B for the adelic point (P_v).

Set $X(\mathbb{A}_k)^{Br} := X(\mathbb{A}_k)^{Br\,X}$,

then: $X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)^{Br}$

$\overline{X(k)}$:= closure of $X(k)$ in $X(\mathbb{A}_k)$.

(direct product of v-adic top.)

Question: For a variety X s.t. $X(\mathbb{A}_k)^{Br} \neq \emptyset$, how to prove $X(k) \neq \emptyset$, or better $\overline{X(k)} = X(\mathbb{A}_k)^{Br}$?

III. 1. Abelian descent

G commutative linear algebraic k-group (ex. G finite, G k-torus).

Consider X-torsors under G.

X-torsor under G := variety Y equipped with a (f.f.) morphism

$$\begin{array}{c} Y \\ f: \;\; \downarrow \\ X \end{array}$$

and an action of G that is

"simply transitive in the geometric fibers":

For $\bar{y}_1, \bar{y}_2 \in Y(\bar{k})$ lying over the same $\bar{x} \in X(\bar{k})$, $\exists!\, g \in G(\bar{k})$, $g \cdot \bar{y}_1 = \bar{y}_2$.

Actually:

Isom. classes of X-torsors under G $\longleftrightarrow$ $H^1(X, G)$

ex: G finite constant, this corresponds to étale Galois coverings with group G (+ action of G given).

Now let $\sigma \in H^1(k, G)$, the twisted torsor Y^σ is defined s.t.

$$[Y^\sigma] = [Y] - [\sigma] \in H^1(X, G).$$

In particular, for $m \in X(k)$:

$$[Y](m) = [\sigma] \iff [Y^\sigma](m) = 0$$

$\iff$ There is a k-point of Y^σ lying over m.

Thus, if you consider the various twisted torsors

$$f^{\sigma} : \begin{array}{c} Y^{\sigma} \\ \downarrow \\ X \end{array}$$

, you obtain:

$$\boxed{X(k) = \bigcup_{\sigma \in H^1(k, G)} f^{\sigma}\left(Y^{\sigma}(k)\right)}$$

Hence: $X(k) \neq \emptyset \iff$ one of the $Y^{\sigma}(k)$ is $\neq \emptyset \Rightarrow Y^{\sigma}(\mathbb{A}_k)$ is not empty for some σ.

<u>Th</u>: The set of $\sigma \in H^1(k, G)$ s.t. $Y^\sigma(\mathbb{A}_k) \neq \emptyset$ is finite.

Define, for the torsor $f : \begin{array}{c} Y \\ \downarrow \\ X \end{array}$

$$X(\mathbb{A}_k)^f := \Big\{ (P_v) \in X(\mathbb{A}_k), \; [Y](P_v) \in \mathrm{Im}\big[H^1(k, G) \to \prod_v H^1(k_v, G)\big] \Big\}$$

$$= \bigcup_{\sigma \in H^1(k, G)} f^\sigma(Y^\sigma(\mathbb{A}_k))$$

Then $\boxed{X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)^f}$

Can do this for any G.
How to find the "meaningful" torsors?

III.2. Link to the BM obstruction

The key point is

Th [Colliot-Thélène / Sansuc, Skorobogatov]

Let $f: \begin{array}{c} Y \\ \downarrow \\ X \end{array}$ be a torsor under a linear commutative group G.

Then

i) $\overline{X(k)} \subset X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)^{f}$

ii) Assume X geom. rational, take G the k-torus s.t. $\hat{G} = \mathrm{Pic}\,\overline{X} = \mathrm{Pic}(X \times_k \bar{k})$.

Then if $f: \begin{array}{c} Y \\ \downarrow \\ X \end{array}$ is a universal torsor under G,

we have $\boxed{X(\mathbb{A}_k)^{Br} = X(\mathbb{A}_k)^{f}}$

Descent method to prove that $X(\mathbb{A}_k)^{Br} \neq \emptyset \Rightarrow X(k) \neq \emptyset$:

a) Write explicit equations for the universal torsors.

b) Try to prove that they satisfy the Hasse principle (Then, since $X(\mathbb{A}_k)^{B} \neq \emptyset$, $\exists\, \sigma$ s.t. $Y^\sigma(\mathbb{A}_k) \neq \emptyset \Rightarrow Y^\sigma(k) \neq \emptyset \Rightarrow X(k) \neq \emptyset$).

Similarly, if the universal torsors satisfy weak approximation, then $\overline{X(k)} = X(\mathbb{A}_k)^{Br}$.

ex: (Colliot-Thélène / Sansuc / Swinnerton-Dyer)

Works for Châtelet surfaces

$$y^2 - az^2 = P(x) \qquad \deg P = 4$$

P irreducible $\Rightarrow Br\,X / Br\,k = 0$
$\Rightarrow$ HP and WA hold.

III.3. Non abelian descent

Let G be a linear alg. k-group (not nec. commutative).

Then, can define $H^0(k, G) = G(k)$

$H^0(X, G) = G(X) = \mathrm{Mor}(X, G)$,

and also $H^1(k, G)$,

$H^1(X, G)$ but these are only pointed sets (not groups)

In this framework, you can consider $f: \begin{array}{c} Y \\ \downarrow \\ X \end{array}$ torsor under G

and descent works essentially as in the abelian case:

$$X(k) \subset \overline{X(k)} \subset X(\mathbb{A}_k)^f.$$

BUT: For G finite non-abelian it is no longer true that

$$X(\mathbb{A}_k)^{Br} \subset X(\mathbb{A}_k)^f$$

Ex [Skorobogatov] $\exists$ bielliptic surface $X/\mathbb{Q}$ with $X(\mathbb{A}_{\mathbb{Q}})^{Br} \neq \emptyset$ and $X(\mathbb{A}_{\mathbb{Q}})^{f} = \emptyset$ for some torsor f under G $\left(G(\bar{\mathbb{Q}}) = (\mathbb{Z}/4)^2 \rtimes \mathbb{Z}/2\right)$.

Th [D. H., 2000]: For any bielliptic surface X over k with $X(k) \neq \emptyset$,

$$\overline{X(k)} \subsetneq X(\mathbb{A}_k)^{Br}$$

Th [D. H., Skorobogatov 2005]: There exists an Enriques surface with $\overline{X(k)} \subsetneq X(\mathbb{A}_k)^{B}$