Counting Rational Points and the Manin Conjecture

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Introductory Workshop on Rational and Integral Points on Higher Dimensional Varieties

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Diophantine Geometry

Diophantus of Alexandria

? 4^{th} Century a.d. ?

Diophantine equations :-

(systems of) polynomial equations over $\mathbb{Z}$ (or $\mathbb{Q}$), to be solved over $\mathbb{Z}$ (or $\mathbb{Q}$)

e.g. $x^n + y^n = z^n$

As hard as any area of math.

Given P, $\exists\, F(x_1, \ldots, x_n)$

"Can one prove P"

equivalent to

"Can one solve $F(x_1, \ldots, x_n) = 0$"

Historically a rag-bag of methods have been used.

Geometric viewpoint - integral/rational points on algebraic varieties.

Geometry over $\mathbb{Q}$ - c.f. Real Algebraic Geometry

Counting points : $V \subseteq \mathbb{A}^n$ or $\mathbb{P}^n$ defined over $\mathbb{Q}$. (Irreducible, degree d)

Height function :

$P = (x_1, \dots, x_n) \in V(\mathbb{Z}) \subseteq \mathbb{A}^n$

$h(P) = \max |x_i|$

$P \in V(\mathbb{Q}) \subseteq \mathbb{P}^n$, $P = (x_0, \dots, x_n)$

$x_i \in \mathbb{Z}$, $h.c.f.(x_0, \dots, x_n) = 1$

$h(P) = \max |x_i|$

$N_V(B) := \# \{ P \in V\left(\begin{smallmatrix}\mathbb{Z}\\ \mathbb{Q}\end{smallmatrix}\right) : h(P) \leq B \}$

Behaviour as $B \to \infty$?

What for?

① Intrinsic interest - enhances the link with geometry of V

(2) Easier problems (?)

Any non-trivial solutions to

$V: x_0^5 + x_1^5 = x_2^5 + x_3^5$?

Estimate $N_V(B)$

(3) Applications elsewhere in number theory.

Waring's Problem. Given $d \in \mathbb{N}$, every sufficiently large integer is a sum of at most $G(d)$ d-th powers.

$G(d) = O(d \log d)$ – Vinogradov

$$V: x_0^d + \cdots + x_{2d-1}^d = x_{2d}^d + \cdots + x_{4d-1}^d$$

If $N_V(B) = O(B^{3d+\varepsilon})$ for each fixed $\varepsilon > 0$, then $G(d) \leq 4d+1$.

Application to the zeros of the Riemann Zeta-function

$$V: \; x_0^h + \cdots + x_{k-1}^h = x_k^h + \cdots + x_{2k-1}^h \quad (1 \le h \le l)$$

Application to the class group structure for ideals of $\mathbb{Q}(\sqrt{d})$

$$V: \quad z^k = x^2 - dy^2$$

What to expect for $N_V(B)$?

$$V: \; F(x_1, \ldots, x_n) = 0 \quad \text{in } \mathbb{A}^n$$

degree d.

$\approx (2B)^n$ choices for $(x_1, \ldots, x_n) = P$

with $h(P) \le B$

$F(x_1, \dots, x_n) \in [-cB^d, cB^d]$ for each

P. "Probability of $F = 0$" $\approx \frac{1}{2cB^d}$

$\therefore$ "Expected size of $N_V(B)$" $\approx c' B^{n-d}$.

Heuristic Expectation

$$B^{n-d} \ll N_V(B) \ll B^{n-d}$$

$(V \subseteq \mathbb{A}^n)$

$$B^{n+1-d} \ll N_V(B) \ll B^{n+1-d}$$

$(V \subseteq \mathbb{P}^n)$

Clearly false if $d > n$ $(d > n+1)$!

Projective Varieties only, from now on.

Theorem (Birch)

Let $V \subseteq \mathbb{P}^n$ be a non-singular hypersurface of degree d. Then $\exists\, c(V)$ s.t.

$$N_V(B) = c(V)\, B^{n+1-d} + o(B^{n+1-d})$$

providing that $n \geq (d-1)2^d$.

Moreover if $n \geq n_0(d)$ then $c(V) > 0$.
(d odd)

Easy cases

$d=1$: $N_V(B) \sim c B^n$

$d=2$: $V(\mathbb{Q})$ can be empty, e.g.

if $V: F(x_0, \ldots, x_n) = 0$ with F positive definite.

V non-singular quadratic, $V(\mathbb{Q}) \neq \emptyset$

then $N_V(B) \sim c B^{n-1}$ as predicted,

as soon as $n \geq 2$: Except when

$n = 3$, and $\det(F) = \square$, in which

case $N_V(B) \sim c B^2 \log B$.

e.g. $x_0 x_1 = x_2 x_3$.

$d \geq 3$ - not easy!

$\dim V = 1$ - curves.

genus zero e.g. $x_0 x_1^{d-1} - x_2^d$

$P = (a^d, b^d, ab^{d-1})$ with $|a|, |b| \leq B^{1/d}$

hcf $(a,b) = 1$

$$B^{2/d} \ll N_V(B) \ll B^{2/d}$$

genus 1. If $V(\mathbb{Q}) \neq \emptyset$.

$$N_V(\mathbb{Q}) \sim c(\log B)^{r/2} \quad \text{(Néron)}$$

$r =$ rank

genus ≥ 2. $V(\mathbb{Q})$ finite (Faltings)

Consistent with the heuristic only for quadric (or linear) curves.

$\dim V = 2$, surfaces.

e.g. $x_0^3 + x_1^3 = x_2^3 + x_3^3$

"trivial" solutions (a, b, a, b) etc.

$$\therefore \quad N_V(B) \gg B^2$$

$$\text{c.f. } B^{n+1-d} = B$$

Here $\{ P = (a,b,a,b) \}$ is a line in V.

Generally "trivial solutions" are those on some fixed proper subvariety of V. Exclude these and consider $\mathcal{U}$ – a Zariski open subset of V.

$$N_{\mathcal{U}}(B) := \#\{ P \in \mathcal{U}(\mathbb{Q}) : h(P) \leq B \}$$

Then, for cubic surfaces, numerical evidence suggests $n+1-d = 3+1-3 = 1$ is the right exponent, if we take $\mathcal{U}$ as the complement of the lines in V.

(Assume V is not a ruled surface !)

One further complication.

When a cubic surface contains two skew lines defined over $\mathbb{Q}$ one can parameterize $V(\mathbb{Q})$. Not particularly helpful!

But $N_U(B) \gg B(\log B)^e$

e.g. $x_0 x_1 x_2 = x_3^3$

some solutions $(a^2b^3, ac^3, d^3, abcd)$

For any C, $B^{1/6} \leq C \leq B^{1/3}$, take

$1 \leq a \leq BC^{-3}$, $1 \leq b \leq B^{-1/3}C^2$

$C/2 \leq c \leq C$, $1 \leq d \leq B^{1/3}$

$\therefore h(a^2b^3, ac^3, d^3, abcd) \leq B$

$\approx B$ solutions.

$C/2 < c \leq C$ so disjoint sets

if C varies over powers of 2

$\therefore N_u(B) \gg B \log B$.

Full analysis $\sim c B (\log B)^6$.

Generally, for cases where the method works,

$$N_u(B) \gg B (\log B)^e$$

e depends on the number of lines, and sets of lines, defined over $\mathbb{Q}$.

Specifically

$$e = \operatorname{rank} \operatorname{Pic}(V) - 1$$

Conjecture, for cubic surfaces,

$$N_U(B) \sim c(V)\, B\, (\log B)^{\text{rank } \mathrm{Pic}(V) - 1}$$

This can be extended to Del Pezzo surfaces, and to Fano varieties in general.

If V is Fano (anticanonical divisor is ample), non-singular, complete intersection $V = W_1 \cap \dots \cap W_t \subseteq \mathbb{P}^n$ of hypersurfaces, $\deg(W_i) = d_i$, with $d_1 + \dots + d_t \leq n$, expect same asymptotic with

$$B^{n+1-d_1-\dots-d_t}$$

Non-singular cubic surfaces :-

Singular cubics

$$X_0 X_1 X_2 = X_3^3$$

$$N_u(B) \sim \frac{B(\log B)^6}{6!} \prod_p \left(1-\frac{1}{p}\right)^7 \left(1+\frac{7}{p}+\frac{1}{p^2}\right)$$

Indeed $B f_6(\log B) + O(B^{1-\delta})$

de la Bretèche

Constant - general conjecture of Peyre

$$X_1 X_2^2 + X_2 X_0^2 + X_3^3 = 0$$

Again $B f_6(\log B) + O(B^{1-\delta})$

de la Bretèche, Browning + Derenthal

$$X_0X_1X_2 + X_0X_1X_3 + X_0X_2X_3 + X_1X_2X_3 = 0$$

$$B(\log B)^6 \ll N_u(B) \ll B(\log B)^6$$

Cayley's cubic — H-B.

Non singular cubic surfaces

$$N_u(B) \gg B(\log B)^{\text{rank Pic}(V)-1}$$

if $\exists$ 2 skew lines / $\mathbb{Q}$

Slater & Swinnerton-Dyer

$$N_u(B) \ll B^{\sqrt{3}+\varepsilon}$$

any fixed $\varepsilon > 0$ - Salberger

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 = 0$$

$$N_U(B) \ll B^{4/3 + \varepsilon}$$

any fixed $\varepsilon > 0$ – H-B.

Methods

First step: Geometric/Elementary

pass to "universal torsor"

divisibility information.

Thus Cayley cubic $\Rightarrow$

$X_i = Z_{ij} Z_{ik} Z_{il} y_j y_k y_l$ $(i, j, k, l \in \{0,1,2,3\}$ distinct$)$

$Z_{ij} = Z_{ji}$ 6 z's 4 y's – 10 variables

$$\sum_{i=0}^{3} y_i z_{jk} z_{jl} z_{kl} = 0$$

$$z_{ik} z_{il} y_j + z_{jk} z_{jl} y_i = z_{ij} v_{ij}$$

$$v_{ij} + v_{kl} = 0 \qquad (3 \text{ variables})$$

$$v_{ij} v_{ik} = z_{il}^2 y_j y_k - z_{jk}^2 y_i y_l$$

Second step – Analytic / Elementary

count solutions of these equations.

These problems bring together those who think geometrically and those who think analytically.