Counting Rational Points and the Manin Conjecture: II

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(A) Sketch proof of the Manin Conjecture for

$$V: \; x_0x_1 - x_2^2 = x_0x_4 - x_1x_2 + x_3^2 = 0$$

(B) General upper bounds for $N_V(B)$.

Theorem (de la Bretèche & Browning) Let $\delta < \frac{1}{12}$. There is a polynomial P, of degree 5 for which

$$N_U(B) = B\,P(\log B) + O(B^{1-\delta}).$$

The degree - 5, and the leading coefficient of P are as predicted by the Manin Conjecture, with Peyre's constant.

[$U = V - \{x_0 = x_2 = x_3 = 0\}$, the unique line in V.]

Weaker version due to Chambert-Loir & Tschinkel.

We'll follow de la Bretèche & Browning

V is a singular Del Pezzo surface of degree 4. Unique singularity $(0,0,0,0,1)$, type D_5

$\mathrm{hcf}(x_0, \dots, x_5) = 1$

$x_0 x_1 = x_2^2$. Set $z_2 := \mathrm{hcf}(x_0, x_1)$

$x_0 = z_2 z_0^2$, $x_1 = z_2 z_1^2$, $x_2 = z_0 z_1 z_2$,
with $\mathrm{hcf}(z_0, z_1) = 1$.
substitute into $x_0 x_4 - x_1 x_2 + x_3^2 = 0$

$$z_0^2 z_2 x_4 - z_0 z_1^3 z_2^2 + x_3^2 = 0$$

so $z_0 z_2 \mid x_3^2$.

Extract the largest square factors

$z_0 = \tilde{z}_0 y_0''^2$, $z_2 = \tilde{z}_2 y_2''^2$

(notation from dlB & B)

$\tilde{Z}_0, \tilde{Z}_2$ no square factor

"square-free"

Set $hcf(\tilde{Z}_0, \tilde{Z}_2) = V_3$,

$\tilde{Z}_0 = V_0 V_3$, $\tilde{Z}_2 = V_2 V_3$; $hcf(V_0, V_2) = 1$

also $hcf(V_0, V_3) = 1$ ($\tilde{Z}_0$ is square-free)

and $hcf(V_2, V_3) = 1$.

$Z_0 Z_2 \mid x_3^2 \Rightarrow V_0 V_2 V_3^2 y_0''^2 y_2''^2 \mid x_3^2$

So $V_3 y_0'' y_2'' \mid x_3$, $V_0 V_2 \mid \left(\dfrac{x_3}{V_3 y_0'' y_2''}\right)^2$.

But V_0, V_2 are square-free & coprime

$\therefore V_0 V_2 \mid \dfrac{x_3}{V_3 y_0'' y_2''}$. $x_3 = V_0 V_2 V_3 y_0'' y_2'' y_3''$,

say.

substitute for z_0, z_2, x_3

divide out $v_0 v_2 v_3^2 y_0''^2 y_2''^2$:-

$$v_0 v_3 y_0''^2 x_4 - v_2 v_3 y_2''^2 z_1^3 + v_0 v_2 y_1''^2 = 0$$

$v_3 \mid v_0 v_2 y_1''^2$. But $hcf(v_0, v_3) = hcf(v_2, v_3) = 1$

and $v_3 \mid \tilde{z}_0$ is square-free. So $v_3 \mid y_1''$

$v_0 \mid v_2 v_3 y_2''^2 z_1^3$ ...... $v_0 \mid y_2''$

$v_2 \mid v_0 v_3 y_0''^2 x_4$ ------ $v_2 \mid y_0''$

$$y_0'' = v_2 y_0', \quad y_2'' = v_0 y_2', \quad y_3'' = v_3 y_3'$$

$$v_2 y_0'^2 x_4 - v_0 y_2'^2 z_1^3 + v_3 y_3'^2 = 0$$

$v_1 := \mathrm{hcf}(y_0', y_2', y_3')$

$y_0' = v_1 y_0 \,, \quad y_2' = v_1 y_2 \,, \quad y_3' = v_1 y_3$

$$\boxed{v_2 y_0^2 x_4 - v_0 y_2^2 z_1^3 + v_3 y_3^2 = 0}$$

Lots of coprimality & square-free-ness conditions

$|x_i| \leq B$

$x_0 = v_0^4 v_1^6 v_2^5 v_3^3 y_0^4 y_2^2$

$x_1 = v_0^2 v_1^2 v_2 v_3 z_1^2 y_2^2$

$x_2 = v_0^3 v_1^4 v_2^3 v_3^2 y_0^2 y_2^2 z_1$

$x_3 = v_0^2 v_1^3 v_2^2 v_3^2 y_0 y_2 y_3$

$x_4 = x_4$

Universal torsor above the minimal desingularisation of V.

Start counting!

Given $v_0, v_1, v_2, \cancel{v_3}, y_0, y_2, y_3, z_1$,

how many possible v_3, x_4?

$r := v_2 y_0^2$, $s := v_0 y_2^2 z_1^3$, $t := y_3^2$

$r x_4 - s + t v_3 = 0$

$|x_0|, \ldots, |x_3| \leq B \Leftrightarrow |v_3| \leq V_3(v_0, \ldots, z_1)$

$\Leftrightarrow x_4 \in I(v_0, \ldots, z_1)$

$J = [-B, B] \cap I$

$\# \{ x_4 \in J : r x_4 \equiv s \ (\mathrm{mod}\ t) \}$

(Pretend that $\mathrm{hcf}(r, t) = 1$)

One solution in each interval of length t.

$\therefore \quad \dfrac{\mathrm{meas}(J)}{t} + O(1)$ solutions

$t = y_3^2$

$$\sum_{v_0, v_1, v_2, y_0, y_2, y_3, z_1} \left\{ \frac{\mathrm{meas}(J(v_0, \dots, z_1))}{y_3^2} + O(1) \right\}$$

Constraints – size, coprimality, square-free.

<u>No algebraic relations</u>

<u>If</u> $y_3^2 = o(\mathrm{meas}(J))$ then

$$\sim \sum \frac{\mathrm{meas}(J)}{y_3^2}$$

Leads to asymptotic relation.

Also lower bounds for $N_U(B)$.

Case y_3 big :-

$r x_4 - s + t v_3 \Rightarrow \quad t v_3 \equiv s \pmod r$

$\# \{ v_3 \in J' : t v_3 \equiv s \pmod r \}$

$\sum \left\{ \frac{\text{meas}(J')}{r} + O(1) \right\}$

OK unless r is too big.

Generally, decompose into cases depending on which variables are large.

A variety of solution counting techniques needed.

Ⓑ General Upper bounds for $N_V(B)$.

Heuristic expectation

$$N_V(B) \approx B^{n+1-d}$$

$V \subseteq \mathbb{P}^n$, a hypersurface, degree d.

True for $d=1$;

$$N_V(B) \ll B^{n+1-2} \log B$$

for $d=2$.

But $N_V(B) \gg B^{n-1}$ if

V contains a $(n-2)$-plane defined over $\mathbb{Q}$ (e.g. a line in a surface)

Conjecture $N_V(B) \ll B^{n-1+\varepsilon}$

for any $\varepsilon > 0$, if $d \geq 2$.

Known for $d \neq 3$; for $d=3$, $n=2,3$ or 4;

for $d=3$, V non-singular

Implied constant depends on d, n, ε

but is uniform in V.

Curves : $n=2$

$N_V(B) \ll B^{2/d+\varepsilon}$ uniform in V

e.g. $x_0 x_1^{d-1} = x_2^d$, $N_V(B) \approx B^{2/d}$

Genus $\geqslant 2 \Rightarrow N_V(B)$ bounded

$N_V(B) \ll 1$ implied constant depends on V.

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Improvement for genus $\geqslant 1$?

Conjecture $N_V(B) \ll_{d,\varepsilon} B^{\varepsilon}$ if V is a curve of genus $\geqslant 1$.

Ellenberg & Venkatesh :- $\exists\, \delta(d) > 0$ s.t.

$N_V(B) \ll_d B^{2/d - \delta(d)}$ for curves V of genus $\geqslant 1$.

Surfaces :

Salberger : $V \subseteq \mathbb{P}^3$ an irreducible surface of degree d. Let U be the open set formed by deleting all curves of degree $< \frac{4\sqrt{d}}{3}$ from V.

Then $N_U(B) \ll B^{3/\sqrt{d} + \varepsilon}$.

If V is non-singular and $d \geqslant 4$ then $U \neq \emptyset$

[$d=3$, can take $U = V - \{\text{lines}\}$]

Roman Surface

$$V: x_0^2 x_1^2 + x_1^2 x_2^2 + x_2^2 x_0^2 = x_0 x_1 x_2 x_3$$

$$N_U(B) \sim c\, B^{3/2}$$

$$U = V - \{\text{lines}\}$$

$\{P \in V(\mathbb{Q}) : h(P) \leq B\}$
$= \{P_1, P_2, \ldots, P_N\}$

$P_i = (x_{0i}, x_{1i}, \ldots, x_{ni})$

Choose D large

$\underline{e} = (e_0, \ldots, e_n)$ $\quad \sum_0^n e_i = D$, $e_i \geq 0$

$e_i \in \mathbb{Z}$.

$\underline{x}^{\underline{e}} := \prod_0^n x_i^{e_i}$

$\#\{\underline{e}\} = m$ say

$M = (\underline{x}_i^{\underline{e}})$ $\quad$ i - rows

$\underline{e}$ - columns

Suppose $m \leq N$

let Δ be a $m \times m$ sub-determinant

Then $|\Delta| \leqslant m!\, B^{mD}$

Key fact: $p^{\nu} \mid \Delta$ for each prime p, with $\nu = \nu(p)$ large.

Thus $\Delta = 0$ if parameters suitably chosen

So $\operatorname{rank}(M) < m$

So $M\underline{c} = \underline{0}$ some $\underline{c} \neq \underline{0}$

$\therefore \sum c_{\underline{e}}\, \underline{x}_i^{\underline{e}} = 0 \quad \forall i$

All points to be counted satisfy

$W: \; C(\underline{x}) = 0.$

Ensure $V \not\subseteq W$

P_i lie on $V \cap W$, a
(when $n=3$)
curve Λ – must control its
degree

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Method generalizes to
higher dimensions
– more work needed

Other approaches – more analytic

- Hardy-Littlewood circle method
- Exponential sums
- Sieves

e.g. Affine $f(x_1, \dots, x_n) = y^2$

$$\#\{(x_1, \dots, x_n ; y) \in \mathbb{Z}^{n+1} : f(x_1, \dots, x_n) = y^2, \max |x_i| \leq B\} = N$$

$\mathcal{P} = \{p \text{ prime}, P < p \leq 2P\}$,

$Q := \# \mathcal{P}$

$$\Big\{ \sum_{p \in \mathcal{P}} \Big(\frac{f(x_1, \dots, x_n)}{p} \Big) \Big\}^2 \geq \begin{cases} 0 & f \neq \square \\ Q^2 & f = \square \end{cases}$$

(ignore $\left(\frac{f}{p}\right) = 0$ for $p \mid f$)

$$N \leq Q^{-2} \sum_{|x_i| \leq B} \Big\{ \sum_p \Big(\frac{f}{p} \Big) \Big\}^2$$

expand $\{\cdots\}^2$

$$\sum_{|x_i| \leq B} \left(\frac{f(x_1, \ldots, x_n)}{p_1 \, p_2} \right)$$

$\ll B^n$ if $p_1 = p_2$

Deligne's exponential/character sum estimates $\Rightarrow$ (essentially)

$\ll (p_1 p_2)^{n/2}$ if $p_1 \neq p_2$

$\therefore N \ll Q^{-2}\{Q B^n + Q^2 P^n\}$

$\ll P^{-1} B^n + P^n$ (almost) Choose $P = B^{n/n+1}$

$N \ll B^{n - \frac{n}{n+1}}$

(c.f. trivial bound B^n)

Lots of scope for variants

$$z^3 = x^2 + d y^2$$

$$h_3(\mathbb{Q}(\sqrt{-d})) \ll d^{\frac{1}{2} - \frac{1}{56} + \varepsilon}$$

(Pierce)

$$z^5 = x^2 + d y^2$$

Finitely many d with $Cl(\mathbb{Q}(\sqrt{-d}))$ elementary abelian exponent 5. (H-B)

Conclusion

Combining geometric and analytic ideas can be very productive, with new applications elsewhere in number theory.