MSRI Summer Graduate Workshop:

Algebraic, Geometric, and Combinatorial Methods for Optimization

Homework Assignments: Week # 1

Instructors: Greg Blekherman, Jiawang Nie

Notation: $H_{n,2d}$ is the vector space of forms (homogeneous polynomials) in n variables of degree $2d$ and $P_{n,2d}$ is the cone of nonnegative polynomials in $H_{n,2d}$.

Problem Set on August 2

1. Let S be the set of all points $(x, y) \in \mathbb{R}^2$ satisfying
$$\begin{bmatrix} 1 & x & x+y \\ x & 1 & y \\ x+y & y & 1 \end{bmatrix} \succeq 0.$$
Draw a picture of this two-dimensional set S. Find the maximum of the linear function x + y over S (Hint: use *YALMIP*).

2. Find the smallest positive real number a such that the following three equations
$$\begin{aligned} x^6 + 1 + ay^2 + az &= 0 \\ y^6 + 1 + az^2 + ax &= 0 \\ z^6 + 1 + ax^2 + ay &= 0 \end{aligned}$$
have a common solution in $\mathbb{R}^3$ (Hint: use *GloptiPoly*).

3. Show that if a polynomial p is nonnegative then its homogenization is nonnegative as well.

4. The *Newton Polytope* N_p of a form p is the convex hull of the vectors of monomial exponents that occur in p. For example, the Newton Polytope of $x^2 + xy + z^2$ is the convex hull of vectors $(2, 0, 0), (1, 1, 0)$ and $(0, 0, 2)$.
Show that if a form $p = \sum_i q_i^2$ is a sum of squares then the Newton Polytope of each q_i is contained in $\frac{1}{2}N_p$. Use this to quickly show that the Motzkin form $M = x^4y^2 + x^2y^4 + z^6 - 3x^2y^2z^2$ is not a sum of squares.

5. Let P be the following convex cone
$$P = \Big\{(f_0, f_1, f_2) : f_0 + f_1x + f_2x^2 \geq 0 \;\forall\, x \in [-1, 1]\Big\}.$$
Find an SDP description for its dual cone P^*.

6. Find the global minimum and minimizer of
$$\begin{aligned} \min_{x\in\mathbb{R}^3} \quad & x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 - (x_1(x_2^2 + x_3^2) + x_2(x_3^2 + x_1^2) + x_3(x_1^2 + x_2^2)) \\ \text{s.t.} \quad & x_1 + x_2 + x_3 - 1 = 0, \\ & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{aligned}$$
(Hint: use *GloptiPoly*).

Problem Set on August 3

7. Let $f(x)$ be the following bivariate polynomial

$$f(x) \;=\; x_1^4 + x_1^2 + x_2^6 - 3x_1^2x_2^2.$$

What is the global minimum f_{min} of this $f(x)$? What is the lower bound f_{sos} returned by the standard SOS relaxation? Is f_{sos} a rational number? If so, give an explicit rational certificate that $f(x) - f_{sos}$ is SOS; if no, give a numerical value of f_{sos}. (Hint: use *SOSTOOLS*).

8. (Parrilo) Consider a univariate polynomial of degree d that is bounded by one in absolute value on the interval $[-1, 1]$. How large can its leading coefficient be? Give an SOS formulation for this problem, and solve it numerically for $d = 2, 3, 4, 5$. Can you guess what the general solution is as a function of d? Can you characterize the optimal polynomial?

9. (Sturmfels) The polynomial $p(x) = 1 + x + x^2 + x^3 + x^4 + x^5 + x^6$ is non-negative on the real line. Find all possible representations of $p(x)$ as a sum of two squares. Does one of these representations involve only rational numbers? Show that the set of all SOS representations of $p(x)$ is a three-dimensional spectrahedron. Draw the picture of this spectrahedron.

10. For a point on the unit sphere $v \in \mathbb{S}^{n-1}$ let l_v denote the following linear functional on $H_{n,2d}$:

$$l_v(p) = p(v) \quad \text{for} \quad p \in H_{n,2d}.$$

Let $K_{n,2d}$ be the conical hull of the functionals l_v in the dual space $H^*_{n,2d}$. Show that $K_{n,2d}$ is the dual cone of $P_{n,2d}$.

11. Show that the form $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$ is nonnegative but not a sum of squares.

12. Show that the maximal faces of $P_{n,2d}$ have the form

$$F_v = \{f \in P_{n,2d} \;\; | \;\; f(v) = 0 \;\;\text{for some}\;\; v \in \mathbb{R}^n\}.$$

Show that the faces F_v have codimension n.

Problem Set on August 5

13. (Sturmfels) For which values of the parameters a and b is the following polynomial non-negative on $\mathbb{R}^2$?

$$f_{a,b}(x,y) \quad = \quad x^4 + y^4 + a(x^3 + y^2) + b(y^3 + x^2) + (a+b).$$

Draw that convex region $\mathcal{C}$ in the (a,b)-plane. The fiber over $(a,b) \in \mathcal{C}$ is the spectrahedron whose points are the SOS representations of $f_{a,b}$. What are the various dimensions of the fibers as (a,b) ranges over the convex set $\mathcal{C}$?

14. (Parrilo) Consider linear maps between symmetric matrices, i.e., of the form $\Lambda : \mathcal{S}^n \to \mathcal{S}^m$. A map is said to be a positive map if it maps the PSD cone $\mathcal{S}^n_+$ into the PSD cone $\mathcal{S}^m_+$ (i.e., it preserves positive semidefinite matrices).

   (a) Show that any linear map of the form $A \mapsto \sum_i P_i^T A P_i$ where each $P_i \in \mathbb{R}^{n\times m}$ is positive. These maps are known as decomposable maps.

   (b) Show that the linear map $C : \mathcal{S}^3 \to \mathcal{S}^3$ (due to M.-D. Choi) given by:

$$C:\ A \mapsto \begin{bmatrix} 2a_{11}+a_{22} & 0 & 0 \\ 0 & 2a_{22}+a_{33} & 0 \\ 0 & 0 & 2a_{33}+a_{11} \end{bmatrix} - A$$

   is a positive map, but is not decomposable.

   Hint: Consider the polynomial defined by $p(x,y) := y^T \Lambda(xx^T)y$. How can you express positivity and decomposability of the linear map Λ in terms of the polynomial p?

15. (Parrilo) Let $M \in \mathcal{S}^n$, and let $z = [x_1^2, \ldots, x_n^2]^T$. The matrix M is said to be copositive if and only if the homogeneous quartic polynomial $p(x) = z^T M z$ is nonnegative.

   (a) Plot the region of $(a,b) \in \mathbb{R}^2$ for which the matrix $\begin{bmatrix} a & b \\ b & 1 \end{bmatrix}$ is copositive.

   (b) Prove that $p(x)$ is a sum of squares if and only if $M = P + N$, where P is positive semidefinite and N is componentwise nonnegative. (Hint: note that $p(x_1,\ldots,x_n) = p(\pm x_1, \ldots, \pm x_n)$).

16. Let C_d be the moment curve parametrized by $(1, t, \ldots, t^d)$ for $t \in \mathbb{R}$. Let K_d be the conical hull of C_d and let P_{2d} be the cone of nonnegative polynomials in 1 variable of degree $2d$ (not homogeneous!). Show that $P_{2d}^* = K_{2d}$.

17. Let F be the set of univariate nonnegative (non-homogeneous) polynomials of degree $2d \geq 4$ such that for all $f \in F$ we have $f(0) = 0$ and the zero has order at least 4. Show that F is a face of P_{2d}. What is the codimension of F? Show that F is a non-exposed face of P_{2d} (Hint: Use the description of the dual cone P_{2d}^*).

18. Find the minimal value of c such that $R + c(x^2+y^2+z^2)^3$ is a sum of squares, where

$$R := x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2$$

Find the exact value of c and provide an exact sos-certificate for it. (The certificate need not be rational, but it needs to be algebraic).

Problem Set on August 6

19. Consider the polynomial optimization:

$$\min_{x \in \mathbb{R}^2} \quad x_1^2 x_2^2 (x_1^2 + x_2^2 - 1).$$

What are the global minimum and minimizers? What is the best lower bound when the standard Lasserre's relaxation is applied? What is the best lower bound when the Jacobian type SDP relaxation is applied? Do they give the same answer? Explain why.

20. Consider the polynomial optimization:

$$\min_{x \in \mathbb{R}^2} \quad 2x_2^4(x_1 + x_2)^4 + x_2^2(x_1 + x_2)^2 + 2x_2(x_1 + x_2) + x_2^2.$$

What is the global infimum? What is the best lower bound when the standard Lasserre's relaxation is applied? What is the best lower bound when the Jacobian type SDP relaxation is applied? Do they give the same answer? Explain why.

21. (Choi-Lam) Show that the zero set of the Robinson form

$$R(x, y, z) = x^6 + y^6 + z^6 - (x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + y^4z^2 + y^2z^4) + 3x^2y^2z^2$$

includes the following 10 projective zeroes: eight given by $[x, y, 1]$ with $x, y \in \{-1, 0, 1\}$ and not both zero, and ninth and tenth at $[1, 1, 0]$ and $[-1, 1, 0]$. Show that the vanishing at these 10 points imposes 27 linearly independent conditions on a nonnegative form $p \in P_{3,6}$ (use your favorite computational package). Conclude that R is an extreme ray of $P_{3,6}$ and it is exposed.

22. Let $v_1, \ldots, v_9$ be the following 9 points in $\mathbb{R}^3$: $(x, y, 1)$ with $x, y \in \{-1, 0, 1\}$. Let $v_1, \ldots, v_4$ be the points with $x, y \neq 0$, $v_5, \ldots, v_8$ be the points with one of x or y equal 0 and v_9 be the point $(0, 0, 1)$. Show that for all ternary cubics p the following relation holds:

$$p(v_1) + p(v_2) + p(v_3) + p(v_4) - 2(p(v_5) + p(v_6) + p(v_7) + p(v_8)) + 4p(v_9) = 0.$$

Use the previous exercise to conclude that the Robinson form is not a sum of squares.

23. (Choi-Lam) Show that the Motzkin form spans an extreme ray of $P_{3,6}$ but it is not exposed.

24. (Delzell) Let $D(x, y, z, w) = w^2 S(x, y, z) + w^8$ with $S(x, y, z) = x^4y^2 + y^4z^2 + z^4x^2 - 3x^2y^2z^2$. Show that $(w^2 + x^2 + y^2 + z^2)^m D$ is not a sum of squares for any $m \geq 0$. (Hint: Assume the opposite and consider the term with highest degree in w).