Inverse Problems

Inverse Problems are problems where causes for a desired or an observed effect are to be determined. They lie at the heart of scientific inquiry and technological development. Applications include a number of medical as well as other imaging techniques, location of oil and mineral deposits in the earth's substructure, creation of astrophysical images from telescope data, finding cracks and interfaces within materials, shape optimization, model identification in growth processes and, more recently, modelling in the life sciences.

The workshop will consist of several 6 minicourses of three or 4 hours each addressing a broad range of the theoretical and practical issues arising in inverse problems including imaging in random media, radar imaging, thermoacoustic and photoacoustic tomography, travel time tomography and electromagnetic imaging. A brief description of each minicourse follows.

- **Imaging in Random Waveguides**

  Lecturer: Liliana Borcea (Rice U.)

  I will discuss the problem of imaging sources/scatterers in random waveguides using measurements of the acoustic pressure field recorded at a remote array of sensors, over some time window. By random waveguides we mean that the wave speed has rapid fluctuations, that cause wave scattering and a significant loss of coherence of the wavefield at the array. I will present an asymptotic theory of wave propagation in such waveguides developed by W. Kohler, G. Papanicolaou and J. Garnier. Then, I will explain how we have used this theory to develop robust imaging methods in random waveguides. In particular, I will present a novel incoherent imaging approach, based on a special form of transport equations arising in the theory mentioned above, to determine in a statistically stable manner the location of sources in the waveguide and statistical information about the wave speed fluctuations.

  References

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  [2] Baggeroer, AB and Kuperman, WA and Mikhalevsky, PN, "An overview of matched field methods in ocean acoustics", IEEE Journal of Oceanic Engineering,vol. 18, no. 4, pp. 401-424, 1993.

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- **Introduction to Radar Imaging**

  Lecturer: Margaret Cheney (RPI)

  Radar imaging is a technology that has been developed, very successfully, within the engineering community during the last 50 years. Radar systems on satellites now make beautiful images of regions of our earth and of other planets. One of the key components of this impressive technology is mathematics, and many of the open problems are mathematical ones.

Reference:

Cheney M. and Borden B., Fundamentals of Radar Imaging, SIAM, 2009

- **An Introduction to Magnetic Resonance Imaging**

  Lecturer Charles Epstein (U. of Pennsylvania)

  This mini-course covers the basic concepts of spin-physics needed to understand the signal equation, and sources of contrast in magnetic resonance imaging, as well as the concepts needed to understand sampling, image reconstruction, the process of selective excitation, and some of the more sophisticated applications of MR-imaging.

  A brief overview of these topics can be found in Introduction to Magnetic Resonance Imaging for Mathematicians, by C.L. Epstein in Ann. Inst. Fourier, 54(2004) 1697-1716. More detailed references are Principles of Nuclear Magnetic Resonance Microscopy, by Paul T. Callaghan, Clarendon Press, Oxford, 1983; Magnetic Resonance Imaging, by E.M. Haacke, R.W. Brown, M.R. Thompson, R. Venkatesan, Wiley-Liss, 1999; and Handbook of MRI Pulse Sequences, by Matt A. Bernstein, Kevin F. King, and Xiaohong Joe Zhou, Elsevier Academic Press, 2004.

  Lecture 1: The physics of spin-1/2 particles in a magnetic field and relaxation mechanisms as described by the phenomenological Bloch equation. The phenomenon of nuclear magnetic resonance, the rotating reference frame and special solutions of the Bloch equation. Faraday's law and the basic signal equation in MR-imaging. The effects of field gradients and the Fourier transform of the spin density. Inversion pulses and the spin echo.

  Lecture 2: Basic properties of the Fourier transform, sampling theory and the discrete Fourier transform. A basic imaging experiment. The problem of selective excitation and its relationship to inverse scattering. Standard imaging pulse sequences and contrast mechanisms. Pulse sequence diagrams.

  Lecture 3: More sophisticated applications of MR-imaging. Topics will be drawn from: The Bloch-Torrey equation and the effects of diffusion; measurement of diffusion constants, and diffusion tensor imaging; flow imaging; parallel imaging; functional imaging.

- **Hybrid Methods of Medical Imaging**

  Lecturer: Peter Kuchment (Texas A&M)

  In traditional tomographic methods, the same physical kind of radiation is used for penetrating the target and for measuring the response (e.g., X-rays in the standard CT, ultrasound in ultrasound tomography, etc.). A variety of new "hybrid" methods is being currently developed, which involve different types of waves. The purpose is to combine the advantages of each type, while alleviating their individual deficiencies. These new modalities usually require new analytic and numerical techniques.

  The lectures will start with a brief review of some novel hybrid methods and then will concentrate on the mathematical techniques and problems of the most developed hybrid modality, the so called thermoacoustic (or photoacoustic) imaging.

  Some hybrid methods are described in [2, 5, 8, **?**, 11]. Surveys of mathematics of thermoacoustic/photoacoustic tomography can be found in [1, 3, 4, 6, 7, 9, 10].

References

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[3] D. Finch and Rakesh. Recovering a function from its spherical mean values in two and three dimensions. In [10], 77-88.

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- 30 **Years of Calderón's Problem**

Lecturer: Gunther Uhlmann (UC Irvine and U. Washington)

In 1980 Calderón published a short paper [3] in which he pioneered the mathematical study of the inverse problem of determining the conductivity of a medium by making voltage and current measurements at the boundary. This inverse method is also called *Electrical Impedance Tomography*. There has been fundamental progress made on this problem, which is now called Calderón's problem, during the following thirty years but several fundamental questions remain unanswered. For a recent survey see [8]. We will consider some of the most important development concentrating in applications of *complex geometrical optics*. We will follow the notes of Mikko Salo and also discuss the two dimensional case [2], [7]. We will also consider counterexamples to uniqueness to Calderón's problem which are relevant for cloaking and invisibility [4],[5].

Time permitting we will give an application of complex geometrical optics to photoacoustics[1]. This inverse problem will be discussed in Peter Kuchment's minicourse.

References

[1] Bal, H., Uhlmann, G., Inverse diffusion theory of photoacoustics, *Inverse Problems*, **26**(2010), 085010.

[2] Bukhgeim, A., Recovering the potential from Cauchy data in two dimensions, *J. Inverse Ill-Posed Probl.*, **16**(2008), 19-34

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http://www.rni.helsinki.fi/~msa/teaching/calderon/calderon_lectures.pdf

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- **Electromagnetic Imaging and the Effect of Small Inhomogeneities**

  Lecturer: Michael Vogelius (Rutgers U.)

  In this set of lectures I shall give a survey of work related to electromagnetic imaging that spans a 20 year period. A first part will be devoted to various representation formulas for the perturbations in the electromagnetic fields caused by volumetrically small sets of inhomogeneities. The imperfections studied range from sets of inhomogeneities consisting of a finite number of well separated objects of known (rescaled) shape and of fixed location, to sets of inhomogeneities of quite "random" geometry and location. The use of these representation formulas to design very effective numerical reconstruction algorithms will also be discussed.

  A second part of the lectures examines the relation between small inhomogeneities and approximate invisibility cloaks. In particular, I shall discuss the use of our representation formulas (or rather, estimates resulting from these formulas) to give quite precise estimates for the degree of approximate invisibility associated with one of these approximate cloaks. Time permitting, I shall touch upon recent approximate invisibility estimates that are also explicit (and sharp) in their dependence on frequency.

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