2013 Summer Graduate Workshop, Cortona, Italy:
Mathematical General Relativity
Schwarzschild Geometry Basics

Let g_E be the Euclidean metric, with Cartesian coordinates $x = (x^1, x^2, x^3)$ so that $g_E = \delta_{ij} dx^i dx^j$, and let $|x| = \sqrt{\sum_{i=1}^{3} (x^i)^2}$. If $r = |x|$, $g_E = dr^2 + r^2 g_{\mathbb{S}^2} = dr^2 + r^2(d\phi^2 + \sin^2(\phi)\, d\theta^2)$. Consider the spatial Schwarzschild metric $g_S = \left(1 + \frac{m}{2|x|}\right)^4 g_E$, defined on the manifold M given by $M = \mathbb{R}^3 \setminus \{0\}$ for $m > 0$, $M = \mathbb{R}^3$ for $m = 0$, and $M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2}\}$ for $m < 0$. Recall that a portion of the maximally extended Schwarzschild space-time $\mathcal{S}$ is given by

$$\bar{g}_S = -\left(\frac{1 - \frac{m}{2|x|}}{1 + \frac{m}{2|x|}}\right)^2 dt^2 + \left(1 + \frac{m}{2|x|}\right)^4 g_E$$

on $|x| > \frac{|m|}{2}$ in case $m \neq 0$. You may use the fact that $\mathrm{Ric}(\bar{g}_S) = 0$, and we can identify $M \subset \mathcal{S} \cap \{t = 0\}$.

Problem 1. a. Show that M is totally geodesic in $\mathcal{S}$.

b. Show that $R(g_S) = 0$ by using the conformal deformation of scalar curvature from the first Problem Set. Show that this is consistent with the Einstein constraint equations.

c. Is $\mathrm{Ric}(g_S) = 0$?

Problem 2. a. For $m > 0$, show that $r \mapsto \frac{m^2}{4r}$ induces an isometry of g_S which fixes $\Sigma_0 = \{r = \frac{m}{2}\}$.

b. For $m > 0$, show that Σ_0 is totally geodesic in M. Express m in terms of the area of Σ_0.

c. Find the area $A(r)$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S. For $m > 0$, show that $A(r)$ has a global minimum at $r = \frac{m}{2}$.

d. When $m < 0$, $A(r) \to 0$ as $r \to -(\frac{m}{2})^+$. Furthermore, a radial geodesic from $r = r_0 > -\frac{m}{2}$ to $r = -\frac{m}{2}$ has finite length. Can the Schwarzschild metric with $m < 0$ be completed by adding in a point?

Problem 3. a. Fix r and find the second fundamental form $I\!I$ and the mean curvature vector $\mathbf{H}$ of $S_r = \{x : |x| = r\}$ of S_r in the metric g_S.

b. Compare $A'(r)$ to $\int_{S_r} \mathbf{H} \cdot \mathbf{X}\, d\sigma$, where $\mathbf{X} = \frac{\partial}{\partial r}$ and $d\sigma$ is the area measure induced by g_S.

c. The *Hawking mass* of a surface Σ is given by

$$m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2\, d\sigma\right).$$

Find $m_H(S_r)$.

Problem 4. Show that there are no closed minimal surfaces in (M, g_S) other than Σ_0 as in Problem 2b. in case $m > 0$. (The argument should follow along the lines of the proof that there are no closed minimal surfaces in $(\mathbb{R}^3, g_E)$.)

Problem 5. Show that

$$m = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{|x|=r} \sum_{i,j=1}^{3} \left((g_S)_{ij,i} - (g_S)_{ii,j}\right) \nu_e^j \, d\sigma_e$$

where the computation is done in the coordinates (x^1, x^2, x^3), and where ν_e is the Euclidean outward unit normal, and $d\sigma_e$ is the Euclidean area measure (where (x^i) are Cartesian coordinates for the Euclidean metric).

Problem 6. In Euclidean space, the spheres minimize surface area for a given enclosed volume V. In fact if a closed surface of area A encloses a volume V, the *isoperimetric inequality* in three dimensions is $V \leq \frac{A^{3/2}}{6\sqrt{\pi}}$.

Let $m > 0$. Hubert Bray showed that the spheres $S_r = \{x : |x| = r\}$ in $(M, (1 + \frac{m}{2r})^4 g_E)$ are isoperimetric in the homology class of Σ_0 (defined above in #2a.). In other words, amongst all surfaces homologous to Σ_0 and enclosing a certain volume V with Σ_0, the one with smallest area is the sphere S_r of the correct r value to enclose volume V.

a. Show that the volume $V(r)$ enclosed by Σ_0 and S_r ($r \geq \frac{m}{2}$) and Σ has the expansion

$$V(r) = \frac{4\pi r^3}{3} \left(1 + \frac{9m}{2r} + O(mr^{-2})\right).$$

b. Conclude that the volume V enclosed by Σ_0 and the sphere S_r of area A has the expansion

$$V(A) = \frac{A^{3/2}}{6\sqrt{\pi}} \left(1 + \frac{(3\sqrt{\pi})m}{\sqrt{A}} + O(mA^{-1})\right).$$

Problem 7. Embedding the Schwarzschild spatial metric.

a. Let $m > 0$. Find an isometric embedding of (M, g_S) into Euclidean space $\mathbb{E}^4$, identified in Cartesian coordinates (x, y, z, w) with $(\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)$. It might be easiest use the other coordinates we introduced for the Schwarzschild metric: $(1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{\mathbb{S}^2}$, $r > 2m$. (This corresponds to "half" of (M, g_S). The map you get will then extend by reflection to the other "half.") For $\omega \in \mathbb{S}^2$, look for an embedding of the form $x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4$. Explain how this justifies the picture we've drawn of the Schwarzschild spatial slice.

b. When $m < 0$ the argument breaks down. Instead, look for an isometric embedding into Minkowski space $\mathbb{M}^4$, which is identified with $\mathbb{R}^4$ with the metric $dx^2 + dy^2 + dx^2 - dw^2$.