Let \( g_E \) be the Euclidean metric, with Cartesian coordinates \( x = (x^1, x^2, x^3) \) so that \( g_E = \delta_{ij}dx^i dx^j \), and let \( |x| = \sqrt{\sum_{i=1}^{3} (x^i)^2} \). If \( r = |x|, g_E = dr^2 + r^2 g_{S^2} = dr^2 + r^2(d\phi^2 + \sin^2(\phi) d\theta^2) \).

Consider the spatial Schwarzschild metric \( g_S = \left(1 + \frac{m^2}{2|x|} \right)^4 g_E \), defined on the manifold \( M \) given by \( M = \mathbb{R}^3 \setminus \{0\} \) for \( m > 0 \), \( M = \mathbb{R}^3 \) for \( m = 0 \), and \( M = \{x \in \mathbb{R}^3 : |x| > -\frac{m}{2} \} \) for \( m < 0 \). Recall that a portion of the maximally extended Schwarzschild space-time \( S \) is given by \( \bar{g}_S = -\left(1 - \frac{m^2}{2|x|} \right)^2 dt^2 + \left(1 + \frac{m}{2|x|} \right)^4 g_E \) on \( |x| > \frac{|m|}{2} \) in case \( m \neq 0 \). You may use the fact that \( \text{Ric}(\bar{g}_S) = 0 \), and we can identify \( M \subset \bar{S} \cap \{t = 0\} \).

**Problem 1.**

a. Show that \( M \) is totally geodesic in \( S \).

b. Show that \( R(g_S) = 0 \) by using the conformal deformation of scalar curvature from the first Problem Set. Show that this is consistent with the Einstein constraint equations.

c. Is \( \text{Ric}(g_S) = 0? \)

**Problem 2.**

a. For \( m > 0 \), show that \( r \mapsto \frac{m^2}{2r} \) induces an isometry of \( g_S \) which fixes \( \Sigma_0 = \{r = \frac{m}{2}\} \).

b. For \( m > 0 \), show that \( \Sigma_0 \) is totally geodesic in \( M \). Express \( m \) in terms of the area of \( \Sigma_0 \).

c. Find the area \( A(r) \) of \( S_r = \{x : |x| = r\} \) of \( S_r \) in the metric \( g_S \). For \( m > 0 \), show that \( A(r) \) has a global minimum at \( r = \frac{m}{2} \).

d. When \( m < 0 \), \( A(r) \to 0 \) as \( r \to -\left(\frac{m}{2}\right)^+ \). Furthermore, a radial geodesic from \( r = r_0 > -\frac{m}{2} \) to \( r = -\frac{m}{2} \) has finite length. Can the Schwarzschild metric with \( m < 0 \) be completed by adding in a point?

**Problem 3.**

a. Fix \( r \) and find the second fundamental form \( II \) and the mean curvature vector \( H \) of \( S_r = \{x : |x| = r\} \) of \( S_r \) in the metric \( g_S \).

b. Compare \( A'(r) \) to \( \int_{S_r} H \cdot X \ d\sigma \), where \( X = \frac{\partial}{\partial r} \) and \( d\sigma \) is the area measure induced by \( g_S \).

c. The Hawking mass of a surface \( \Sigma \) is given by

\[
m_H(\Sigma) = \sqrt{\frac{A(\Sigma)}{16\pi}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2 \ d\sigma \right).
\]

Find \( m_H(S_r) \).
Problem 4. Show that there are no closed minimal surfaces in \((M,g_S)\) other than \(\Sigma_0\) as in Problem 2b. in case \(m > 0\). (The argument should follow along the lines of the proof that there are no closed minimal surfaces in \((\mathbb{R}^3,g_E)\).)

Problem 5. Show that

\[
m = \frac{1}{16\pi} \lim_{r \to +\infty} \int_{|x|=r} \sum_{i,j=1}^{3} ((g_S)_{ij,i} - (g_S)_{ii,j}) \nu_e^i \, d\sigma_e
\]

where the computation is done in the coordinates \((x^1, x^2, x^3)\), and where \(\nu_e\) is the Euclidean outward unit normal, and \(d\sigma_e\) is the Euclidean area measure (where \((x^i)\) are Cartesian coordinates for the Euclidean metric).

Problem 6. In Euclidean space, the spheres minimize surface area for a given enclosed volume \(V\). In fact if a closed surface of area \(A\) encloses a volume \(V\), the isoperimetric inequality in three dimensions is \(V \leq \frac{A^{3/2}}{6\sqrt{\pi}}\).

Let \(m > 0\). Hubert Bray showed that the spheres \(S_r = \{ x : |x| = r \}\) in \((M,(1 + \frac{m^2}{2r})^4g_E)\) are isoperimetric in the homology class of \(\Sigma_0\) (defined above in #2a.). In other words, amongst all surfaces homologous to \(\Sigma_0\) and enclosing a certain volume \(V\) with \(\Sigma_0\), the one with smallest area is the sphere \(S_r\) of the correct \(r\) value to enclose volume \(V\).

a. Show that the volume \(V(r)\) enclosed by \(\Sigma_0\) and \(S_r\) \((r \geq \frac{mr}{2})\) and \(\Sigma\) has the expansion

\[
V(r) = \frac{4\pi r^3}{3} \left( 1 + \frac{9m}{2r} + O(mr^{-2}) \right).
\]

b. Conclude that the volume \(V\) enclosed by \(\Sigma_0\) and the sphere \(S_r\) of area \(A\) has the expansion

\[
V(A) = \frac{A^{3/2}}{6\sqrt{\pi}} \left( 1 + \frac{(3\sqrt{\pi})m}{\sqrt{A}} + O(mA^{-1}) \right).
\]

Problem 7. Embedding the Schwarzschild spatial metric.

a. Let \(m > 0\). Find an isometric embedding of \((M,g_S)\) into Euclidean space \(\mathbb{E}^4\), identified in Cartesian coordinates \((x,y,z,w)\) with \((\mathbb{R}^4, dx^2 + dy^2 + dz^2 + dw^2)\). It might be easiest use the other coordinates we introduced for the Schwarzschild metric: \((1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{S^2}, r > 2m\). (This corresponds to “half” of \((M,g_S)\). The map you get will then extend by reflection to the other “half.”) For \(\omega \in S^2\), look for an embedding of the form \(x = r\omega \mapsto (r\omega, \xi(r)) \in \mathbb{R}^4\). Explain how this justifies the picture we’ve drawn of the Schwarzschild spatial slice.

b. When \(m < 0\) the argument breaks down. Instead, look for an isometric embedding into Minkowski space \(\mathbb{M}^4\), which is identified with \(\mathbb{R}^4\) with the metric \(dx^2 + dy^2 + dx^2 - dw^2\).