1. \( \bar{\partial} \) method and Hartogs extension

We begin in complex euclidean space \( \mathbb{C}^n \), with standard coordinates \((z_1, \ldots, z_n)\), \( z_k \in \mathbb{C} \) and \( z_k = x_k + iy_k \) for \( k = 1, \ldots, n \).

1.1. Cauchy-Riemann complex I. Let \( \Omega \subset \mathbb{C}^n \) be a domain, i.e. an open, connected set. If \( f : \Omega \to \mathbb{C} \) and \( f \in C^1(\Omega) \) (see Function Space table), define

\[
\frac{\partial f}{\partial z_k} = \frac{1}{2} \left[ \frac{\partial f}{\partial x_k} - i \frac{\partial f}{\partial y_k} \right], \quad \frac{\partial f}{\partial \bar{z}_k} = \frac{1}{2} \left[ \frac{\partial f}{\partial x_k} + i \frac{\partial f}{\partial y_k} \right], \quad k = 1, \ldots, n.
\]

(1.1)

The equations (1.1) define the first-order differential operators \( \frac{\partial}{\partial z_k} \) and \( \frac{\partial}{\partial \bar{z}_k} \). These operators are called the Cauchy-Riemann vector-fields and are easily seen to be derivations; thus we also refer to these operators as tangent vectors on \( \mathbb{C}^n \). If we consider the coordinate functions \( e_k(z_1, \ldots, z_n) = z_k \) and their conjugates \( \bar{e}_k(z_1, \ldots, z_n) = \bar{z}_k \), notice

\[
\frac{\partial}{\partial z_k}(e_\ell) = \delta_{kl}, \quad \frac{\partial}{\partial \bar{z}_k}(\bar{e}_m) = 0,
\]

where \( \delta_{kl} \) is the Kronecker symbol, while

\[
\frac{\partial}{\partial z_k}(\bar{e}_m) = 0 = \frac{\partial}{\partial \bar{z}_k}(e_m), \quad 1 \leq k, m \leq n.
\]

Differential forms, or co-tangent vectors, connected to the complex structure are also important. The basic first-order ones are

\[
dz_k = dx_k + idy_k \quad \text{and} \quad d\bar{z}_k = dx_k - idy_k, \quad k = 1, \ldots, n.
\]

Tangent vectors and co-tangent vectors naturally “pair” with each other. This is elaborated later. For now, take the view that \( \{dz_k, d\bar{z}_k\} \) are basis elements of an abstract vector space. When a function is written adjacent to one of these symbols, the understanding is the function is evaluated at a point and this number multiplies the corresponding symbol. Since the evaluation is often not explicitly written, this may be confusing initially but becomes familiar with practice.

For \( f \in C^1(\Omega) \), the exterior derivative of \( f \) is defined

\[
df = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j + \sum_{j=1}^n \frac{\partial f}{\partial y_j} dy_j.
\]
Some algebra allows this to be re-written as
\[ df = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j + \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j} d\bar{z}_j \]
\[ = \partial f + \bar{\partial} f. \tag{1.2} \]

The last equation defines the operators \( \partial \) and \( \bar{\partial} \), i.e.,
\[ \bar{\partial} f(z) = \sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}(z) d\bar{z}_j \quad \text{for } z \in \Omega \]

and similarly for \( \partial f \). Thus \( \bar{\partial} f \) is a linear combination of the differential forms \( d\bar{z}_j \). In this way, it is an example of a \((0,1)\)-form.

1.2. Holomorphic functions.

**Definition 1.3.** Let \( \Omega \subset \mathbb{C}^n \) be a domain. A function \( f \in C^1(\Omega) \) is **holomorphic** on \( \Omega \) if \( \bar{\partial} f(z) = 0 \) for all \( z \in \Omega \). Let \( \mathcal{O}(\Omega) \) denote the set of holomorphic functions on \( \Omega \).

It’s important to remember that \( \bar{\partial} f(z) = 0 \), which looks like a scalar equation, is actually a system of equations: \( \bar{\partial} f = 0 \) means \( \frac{\partial f}{\partial \bar{z}_j} = 0 \) for all \( j = 1, \ldots, n \).

The equation \( \frac{\partial f}{\partial \bar{z}_j}(z) = 0 \), for some \( j \), says that \( f \) satisfies the one-variable Cauchy-Riemann equations in the complex variable \( z_j \). Exercise I. Since we are assuming \( f \in C^1(\Omega) \), it follows that \( f \) is holomorphic in \( z_j \). Thus Definition 1.3 says every \( f \in \mathcal{O}(\Omega) \) is holomorphic as a function of one complex variable separately in each of the underlying variables \( z_1, \ldots, z_n \).

A theorem of Hartogs says that a function \( f(z_1, \ldots, z_n) \) that is holomorphic separately in each \( z_j \) must also satisfy Definition 1.3 i.e. the condition \( f \in C^1(\Omega) \) necessarily follows. This is false for smooth functions as the example \( g(x, y) = \frac{xy}{x^2+y^2} \) in \( \mathbb{R}^2 \) shows. The proof of Hartogs theorem is outside the main theme of these lectures, but will be discussed in the afternoon sessions.

1.3. Cauchy-Riemann complex II. A \((p, q)\)-form, for \( p \) and \( q \) positive integers such that \( 1 \leq p, q \leq n \), is obtained by taking wedge products of the differentials \( dz_k \) and \( d\bar{z}_k \), with exactly \( p \) factors of the various \( dz_k \)s and \( q \) factors of the \( d\bar{z}_k \)s. Multi-index notation is useful: if \( I = (i_1, \ldots, i_p) \) and \( J = (j_1, \ldots, j_q) \) are \( p \) and \( q \)-tuples respectively, write

\[ dz^I = dz_{i_1} \wedge \cdots \wedge dz_{i_p} \quad \& \quad d\bar{z}^J = d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q}, \]


and denote \( p = |I|, q = |J| \).

A \((p, q)\)-form is a linear combination of \( dz^I \wedge d\bar{z}^J \), as \( I \) and \( J \) range over all indices such that \(|I| = p \) and \(|J| = q \), i.e.

\[ \beta = \sum_{|I|=p \atop |J|=q} \beta_{IJ} dz^I \wedge d\bar{z}^J \tag{1.4} \]

is a \((p, q)\)-form, where the coefficients \( \beta_{IJ} \) are functions on \( \Omega \).

Classes of \((p, q)\)-forms are obtained by requiring each \( \beta_{IJ} \) to belong to certain function spaces. Initially the most relevant classes of \((p, q)\)-forms are those with coefficients in \( C^\infty(\Omega) \), \( C^\infty(\overline{\Omega}) \), or \( C^\infty_0(\Omega) \), which we’ll denote as

\[ \Lambda^{p,q}(\Omega), \quad \Lambda^{p,q}(\overline{\Omega}), \quad \text{or} \quad \Lambda^{p,q}_0(\Omega) \]

respectively.
1.4. The $\bar{\partial}$ method. The main analytic problem in several complex variables is the construction of functions $h \in \mathcal{O}(\Omega)$ with various side properties $\mathcal{P}$. There are many interesting $\mathcal{P}$, some examples are: taking given values on prescribed sub-varieties of $\Omega$, blowing up as $z \to p \in b\Omega$, and being metrically or norm dominated by a given expression.

Power series methods, which are very successful in one-variable, generally fail in $\mathbb{C}^n, n \geq 2$. A powerful several variable technique for construction in several variables is to consider the inhomogeneous $\bar{\partial}$ system.

The method:

(a) Construct $f \in C^\infty(\Omega)$ with the desired property $\mathcal{P}$ — this is usually easy.
(b) Consider the $(0,1)$-form $\alpha := \bar{\partial} f$.
(c) Find a solution to $\bar{\partial} u = \alpha$ that is well-behaved wrt property $\mathcal{P}$.
(d) Define $h := f - u$. Then $h \in \mathcal{O}(\Omega)$ and satisfies $\mathcal{P}$.

The italicized phrase suggests both the scope and challenge of the method. Its precise formulation depends on the character of property $\mathcal{P}$. We turn to our first example in the next section.

1.4.1. $\bar{\partial}$ is over-determined. The method requires solving $\bar{\partial} u = \alpha$, for $\alpha$ a given $(0,1)$-form. Or, in more general situations, if $\alpha$ is a $(p,q)$-form. For the moment, consider forms with smooth coefficients, i.e., in $\Lambda^{p,q}(\Omega)$, to focus on the main issue. If $\alpha \in \Lambda^{0,1}(\Omega)$, the equation $\bar{\partial} u = \alpha$ is over-determined in $\mathbb{C}^n, n > 1$.

$$\bar{\partial} u = \alpha \iff \begin{cases} \frac{\partial u}{\partial \bar{z}_1}(z) = \alpha_1(z) \\ \vdots \\ \frac{\partial u}{\partial \bar{z}_n}(z) = \alpha_n(z) \end{cases}$$

Notice there are $n$ data functions $\alpha_1, \ldots, \alpha_n$ — but only a single solution $u$. Therefore $\alpha$ must satisfy compatibility conditions before there is a chance to solve $\bar{\partial} u = \alpha$. Indeed if $u$ solves $\bar{\partial} u = \alpha$, then

$$\frac{\partial \alpha_k}{\partial \bar{z}_j} = \frac{\partial^2 u}{\partial \bar{z}_j \partial \bar{z}_k} = \frac{\partial \alpha_j}{\partial \bar{z}_k}, \quad \forall k, j.$$ (1.5)

Consequently only $\alpha$ satisfying the necessary conditions (1.5) are allowed when trying to solve $\bar{\partial} u = \alpha$. The conditions (1.5) are more succinctly expressed after the $\bar{\partial}$ operator is extended to higher level forms in Lecture 3.

1.5. Solving $\bar{\partial}$ with compact support. Our first example of an interesting side condition is to take $\mathcal{P} =$ “has compact support”.

**Theorem 1.6** (Solving $\bar{\partial}$ with compact support). If $\alpha \in \Lambda^{0,1}_0(\mathbb{C}^n), n > 1$, satisfies conditions (1.5), there exists $u \in \Lambda^{0,0}_0(\mathbb{C}^n)$ such that $\bar{\partial} u = \alpha$.

**Remark 1.7.** This result does not hold in $\mathbb{C}^1$, unless $\int_{\mathbb{C}} \alpha(\zeta) \zeta^k dV(\zeta) = 0$ for all $k \in \mathbb{Z}^+$. 

**Exercise II.** The proof of Theorem 1.6 uses the one-variable generalized Cauchy Integral formula. This result is important enough to state separately.
Theorem 1.8 (Generalized CIF). If $\Omega \subset \mathbb{C}$ is a domain with $C^1$ boundary (see Lecture 5) and $u \in C^1(\Omega)$, then
\begin{equation}
 u(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{u(\zeta)}{\zeta - z} \, d\zeta + \frac{1}{2\pi i} \int_{\Omega} \frac{\partial u / \partial \zeta}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}, \quad z \in \Omega. \tag{1.9}
\end{equation}

Proof. Fix $z \in \Omega$. For $\epsilon > 0$, let $B(z, \epsilon) = \{w \in \mathbb{C} : |w - z| < \epsilon\}$ be the disc centered at $z$ of radius $\epsilon$. Consider the domain $\Omega_\epsilon = \Omega \setminus B(z, \epsilon)$.

Recall Stokes theorem: if $D$ is a $C^1$ bounded domain and $\omega \in \Lambda^{1,0}(\overline{D})$ then
\[ \int_{\partial D} \omega = \int_{D} d\omega, \]
where $bD$ is positively oriented. Apply Stokes to $\omega(\zeta) = \frac{u(\zeta)}{\zeta - z} \, d\zeta$ on the domain $\Omega_\epsilon$, noting the two separate components, $b\Omega$ and $bB(z, \epsilon)$, of the boundary $b\Omega_\epsilon$. The result is
\[ \int_{\partial \Omega_\epsilon} \frac{u(\zeta)}{\zeta - z} \, d\zeta - i \int_{0}^{2\pi} u\left(z + \epsilon e^{i \theta}\right) \, d\theta = \int_{\Omega_\epsilon} \frac{\partial u / \partial \zeta}{\zeta - z} \, d\zeta \wedge d\bar{\zeta}. \]
The second integral on the LHS is $(b\mathbb{B}(z, \epsilon)) \omega$, parameterizing by $\zeta(\theta) = z + \epsilon e^{i \theta}$ and the $−$ sign coming from positively orienting $b\Omega_\epsilon$. Rearranging this equation, using $d\zeta \wedge d\zeta = -d\zeta \wedge \bar{\zeta}$,
\[ \int_{\partial \Omega_\epsilon} \frac{u(\zeta)}{\zeta - z} \, d\zeta + \int_{\Omega_\epsilon} \frac{\partial u / \partial \zeta}{\zeta - z} \, d\zeta \wedge d\bar{\zeta} = i \int_{0}^{2\pi} u\left(z + \epsilon e^{i \theta}\right) \, d\theta. \]

Let $\epsilon \to 0$. The RHS tends to $2\pi i u(z)$, since $u$ is continuous. This gives (1.9). □

Note that when $u \in O(\Omega)$, Theorem 1.8 reduces to the usual Cauchy Integral formula.

Proof of Theorem 1.6 Define
\[ u(z_1, z_2, \ldots, z_n) = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\alpha_1(\zeta, z_2, \ldots, z_n)}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta}, \]
\[ = \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\alpha_1(\zeta + z_1, z_2, \ldots, z_n)}{\zeta} \, d\zeta \wedge d\bar{\zeta}. \]
The second equality follows by change of variables $\zeta \to \zeta + z_1$.

Consider the second integral defining $u$. Differentiation under the integral sign shows $u \in C^\infty(\mathbb{C})$. Additionally, taking $\frac{\partial}{\partial z_k}$, for any $k = 1, \ldots, n$, of both sides yields
\begin{align}
\frac{\partial u}{\partial z_k} &= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \alpha_1}{\partial z_k}(\zeta + z_1, z_2, \ldots, z_n) \frac{1}{\zeta} \, d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \alpha_1}{\partial z_k}(\zeta + z_1, z_2, \ldots, z_n) \frac{1}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta} \\
&= \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{\partial \alpha_k}{\partial z_1}(\zeta, z_2, \ldots, z_n) \frac{1}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta}, \tag{1.10}
\end{align}
The compatibility conditions (1.5) give the second equality. The third equality follows by change of variables.

Choose $R$ large enough so that $\text{supp} \, \alpha \subset B(0, R)$. In particular $\alpha \equiv 0$ on $bB(0, R)$. Theorem 1.8 then says that
\[ \frac{1}{2\pi i} \int_{B(0, R)} \frac{\partial \alpha_k}{\partial z_1}(\zeta, z_2, \ldots, z_n) \frac{1}{\zeta - z_1} \, d\zeta \wedge d\bar{\zeta} = \alpha_k(z_1, \ldots, z_n), \]
But the LHS above equals the RHS of (1.10). Thus \( \frac{\partial u}{\partial \bar{z}_k} = \alpha_k \), \( k = 1, \ldots, n \); in other words \( \partial u = \alpha \).

\[ \square \]

1.6. **Hartogs extension.** Theorem [1.6] leads to a remarkable extension result proved by Hartogs (by a different method) more than 100 years ago. The phenomena expressed by this result has no analog in \( \mathbb{C}^1 \). This result (and the biholomorphic inequivalence of topological cells in \( \mathbb{C}^n \)) inaugurated the study of several complex variables as a separate field.

**Theorem 1.11 (Hartogs).** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \), \( n > 1 \), and \( K \subset \subset \Omega \) such that \( \Omega \setminus K \) is connected.

Then every \( f \in \mathcal{O}(\Omega \setminus K) \) is the restriction of an \( F \in \mathcal{O}(\Omega) \).

Let’s sketch how the \( \partial \) method works here:

(a) Let \( \chi \in C_0^\infty(\Omega) \) such that \( \chi \equiv 1 \) on a neighborhood of \( K \). Set

\[
 f(z) = \begin{cases} 
 (1 - \chi) \cdot f(z) & \text{if } z \in \Omega \setminus K \\
 0 & \text{if } z \in K 
\end{cases}
\]

(b) Set \( \alpha = \bar{\partial} f \). Note that \( \alpha = - (\bar{\partial} \chi) \cdot f \) where \( f \) is defined, \( \bar{\partial} \alpha = 0 \), and \( \alpha \in \Lambda_0^{0,1}(\mathbb{C}^n) \) (just extend \( \alpha \) by 0 off \( \text{supp}(\bar{\partial} \chi) \)).

(c) Invoke Theorem 1.6 to find \( u \in \Lambda_0^{0,0}(\mathbb{C}^n) \) solving \( \bar{\partial} u = \alpha \).

(d) Define \( F = f - u \). Consider the component of \( \partial \Omega \) that also bounds a neighborhood of \( \infty \). In an open set of this boundary component, \( u \equiv 0 \) and \( \chi \equiv 0 \). Thus, \( F = f \) on this set. Since \( \Omega \setminus K \) is connected, \( F = f \) on all of \( \Omega \setminus K \).

A more formal proof of Theorem 1.11 can be found in Krantz, “Function theory of several complex variables”, Theorem 1.2.6.