2. Basic properties of $O(\Omega)$

In this lecture we will study the standard local properties of holomorphic functions using the one variable results and calculus of several variables.

The space $\mathbb{C}^n$ can be identified with $\mathbb{R}^{2n}$ in the following sense. Given $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$, each coordinate can be written as $z_j = x_j + iy_j$, with $x_j, y_j \in \mathbb{R}$. The mapping

$$z \mapsto (x_1, y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n}$$

establishes an $\mathbb{R}$-linear isomorphism between $\mathbb{C}^n$ and $\mathbb{R}^{2n}$, which is compatible with the metric structures.

The open ball of radius $r > 0$ centered at $a \in \mathbb{C}^n$ is defined by

$$B(a, r) = \{z \in \mathbb{C}^n : |z - a| < r\}.$$

In several complex variables, it is often convenient to use another system of neighborhoods: the open polydiscs.

**Definition 2.1 (Polydisc).** An open polydisc centered at $a = (a_1, \ldots, a_n) \in \mathbb{C}^n$ and of polyradius $r = (r_1, \ldots, r_n)$, where $r_j \geq 0$ for all $0 \leq j \leq n$, is given by the set

$$P(a, r) = \{z \in \mathbb{C}^n : |z_j - a_j| < r_j \text{ for each } 0 \leq j \leq n\}.$$

It is quite obvious from the definition that an open polydisc is a Cartesian product of $n$ planar open discs.

**2.1. Cauchy Integral Formula on Polydiscs.** A lot of basic local properties of holomorphic function in one variable follows from the Cauchy Integral Formula which generalizes to polydiscs quite easily:

**Theorem 2.2 (CIF on polydiscs).** Let $P = P(a, r)$ be a polydisc in $\mathbb{C}^n$ and let $f \in C(\overline{P}) \cap O(P)$. Then

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b_0 P} \frac{f(\zeta_1) \ldots d\zeta_n}{(\zeta_1 - z_1) \ldots (\zeta_n - z_n)} \text{ for } z \in P,$$

where $b_0 P = \{\zeta \in \mathbb{C}^n : |\zeta_j - a_j| = r_j, 1 \leq j \leq n\}$.

**Remark 2.4.** It is important to note that the integral in not over the entire boundary of the polydisc. This part of the boundary $b_0 P$ is called the distinguished boundary of the polydisc.

**Proof.** For simplicity, we will prove this for $n = 2$. By Cauchy Integral Formula for one variable, if we fix $z_2$ such that $|z_2 - a_2| < r_2$, we have

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{\zeta_1 - a_1 = r_1} \frac{f(\zeta_1, z_2)}{(\zeta_1 - z_1)} d\zeta_1$$
for any \( z_1 \) with \(|z_1 - a_1| < r_1 \). Similarly for each fixed \( \zeta_1 \),
\[
f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1 - a_1| = r_1} \int_{|\zeta_2 - a_2| = r_2} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_2 d\zeta_1
\]
for each \((z_1, z_2) \in P\). The result for general \( n \) can be done using induction on \( n \). \( \square \)

Exercise I: Show that if \( z \in \overline{P} \) we have \(|f(z)| \leq |f|_{L^\infty(b_0P)} \) for all \( f \in \mathcal{O}(P) \cap C(\overline{P}) \).

**Theorem 2.5** (Cauchy estimates). Let \( f \in \mathcal{O}(P(a, r)) \). Then for all \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n \),
\[
|D^\alpha f(a)| \leq \frac{\alpha!}{r^n} \|f\|_{L^\infty(P(a, r))}; \quad (2.6)
\]
\[
|D^\alpha f(a)| \leq \frac{\alpha!(\alpha_1 + 1)\cdots(\alpha_n + 1)}{(2\pi)^n r^{\alpha + 2}} \|f\|_{L^1(P(a, r))}. \quad (2.7)
\]
Here \( D^\alpha = \frac{\partial^{\alpha_1 + \cdots + \alpha_n}}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} \), \( \alpha! = \alpha_1! \cdots \alpha_n! \), \( r^\alpha = r_1^{\alpha_1} \cdots r_n^{\alpha_n} \) and for \( m \in \mathbb{Z} \), \( \alpha + m = (\alpha_1 + m, \ldots, \alpha_n + m) \).

**Proof.** Fix \( 0 < \rho < r \). Applying Theorem 2.2 to \( P(a, \rho) \subset P(a, r) \) and differentiating under the integral sign, we get
\[
D^\alpha f(a) = \frac{\alpha!}{(2\pi i)^n} \int_{b_0P(a, \rho)} \frac{f(\zeta)}{(\zeta - a)^{\alpha + 1}} d\zeta. \quad (2.8)
\]
After an obvious estimation of (2.8) and taking \( \lim_{\rho \to r} \) both sides we get (2.6). For (2.7), change to polar coordinates, multiply both sides of (2.8) by \( \rho^{\alpha + 1} \), and taking the estimate gives
\[
|D^\alpha f(a)| \rho^{\alpha + 1} \leq \frac{\alpha!}{(2\pi)^n} \int_{|0, 2\pi|^n} |f(a + pe^{i\theta})| \rho_1 \cdots \rho_n d\theta_1 \cdots d\theta_n. \quad (2.9)
\]
Integrating (2.9) both sides over \( 0 \leq \rho_j \leq r_j \) for \( 1 \leq j \leq n \) gives (2.7). \( \square \)

Exercise I: Let \( \{f_j\}_{j=1}^\infty \subset \mathcal{O}(\Omega) \) be a sequence of holomorphic functions converging compactly in \( \Omega \) to \( f \). Then prove that \( f \in \mathcal{O}(\Omega) \), and for each \( \alpha \in \mathbb{N}^n \), \( \lim_{j \to \infty} D^\alpha f_j = D^\alpha f \).

### 2.2. Local power series expansion.

**Definition 2.10.** The multiple series \( \sum_{\nu \in \mathbb{N}^n} b_\nu \) is called absolutely convergent if
\[
\sum_{\nu \in \mathbb{N}^n} |b_\nu| = \sup \left\{ \sum_{\nu \in \Lambda} |b_\nu| : \Lambda \text{ finite} \right\} < \infty.
\]
A power series in \( n \) complex variables \( z_1, \ldots, z_n \) centered at \( a \in \mathbb{C}^n \) is a multiple series \( \sum_{\nu \in \mathbb{N}^n} b_\nu \) with
\[
b_\nu = c_\nu (z - a)^\nu = c_{\nu_1 \cdots \nu_n} (z_1 - a_1)^{\nu_1} \cdots (z_n - a_n)^{\nu_n},
\]
where \( c_\nu \in \mathbb{C} \) for \( \nu \in \mathbb{N}^n \). We will usually consider the power series centered at the origin.

**Definition 2.11.** The domain of convergence \( \Omega \) of the power series
\[
\sum_{\nu \in \mathbb{N}^n} c_\nu z^\nu \quad (2.12)
\]
is the interior of the set of points \( z \in \mathbb{C}^n \) for which (2.12) converges absolutely.
**Lemma 2.13** (Abel’s lemma). Suppose \( c_\nu \in \mathbb{C} \) for \( \nu \in \mathbb{N}^n \) and that for some \( w \in \mathbb{C}^n \)
\[
\sup_{\nu \in \mathbb{N}^n} |c_\nu w^\nu| = M < \infty.
\] (2.14)
Let \( r = \tau(w) = (|w_1|, \ldots, |w_n|) \). Then the power series \( \sum c_\nu z^\nu \) converges on the polydisc \( P(0, r) \). Moreover, the convergence is **normal** in the following sense: if \( K \subset P(0, r) \) is compact and \( \epsilon > 0 \) is arbitrary, there is a finite set \( \Lambda = \Lambda(K, \epsilon) \), such that
\[
\sum_{\nu \in \Lambda} |c_\nu z^\nu| < \epsilon \quad \text{for all} \quad z \in K.
\]

**Proof.** To be given in Lecture 4. \( \square \)

**Theorem 2.15** (Taylor series). Let \( f \in \mathcal{O}(P(a, r)) \). Then the taylor series of \( f \) converges to \( f \) absolutely and uniformly on compact subsets of \( P(a, r) \), i.e.,
\[
f(z) = \sum_{\nu \in \mathbb{N}^n} \frac{D^\nu f(a)}{\nu!} (z - a)^\nu \quad \text{for} \quad z \in P(a, r).
\]

**Proof.** From the Cauchy integral formula (2.3), applied to \( z \in P(a, \rho) \subset P(a, r) \), we expand \((\zeta - z)^{-1} = (\zeta_1 - z_1)^{-1} \cdots (\zeta_n - z_n)^{-1}\) into multiple geometric series
\[
(\zeta - z)^{-1} = \sum_{\nu \in \mathbb{N}^n} \frac{(z - a)^\nu}{(\zeta - a)^{\nu + 1}}, \tag{2.16}
\]
which converges uniformly for \( \zeta \in b_0 P(a, \rho) \), since \(|z_j - a_j|/|\zeta_j - a_j| \leq |z_j - a_j|/\rho_j < 1 \) for such \( \zeta \) and for all \( 1 \leq j \leq n \). Substituting (2.16) to (2.3) and interchanging summation and integration, we get
\[
f(z) = \sum_{\nu \in \mathbb{N}^n} \left[ \frac{1}{(2\pi i)^n} \int_{b_0 P(a, \rho)} \frac{f(\zeta)d\zeta_1 \cdots d\zeta_n}{(\zeta - a)^{\nu + 1}} \right] (z - a)^\nu \tag{2.17}
\]
for \( z \in P(a, \rho) \). By (2.8), the coefficient of \((z - a)^\nu\) in (2.17) equals \( D^\nu f(a)/\nu! \). \( \square \)

**Theorem 2.18** (The identity theorem). Let \( \Omega \subset \mathbb{C}^n \) be a domain (open and connected) and let \( f \in \mathcal{O}(\Omega) \). If \( a \in \Omega \) is such that \( D^\alpha f(a) = 0 \) for all \( \alpha \in \mathbb{N}^n \), then \( f(z) = 0 \) for \( z \in \Omega \). In particular, if there is a nonempty open set \( U \subset \Omega \), such that \( f(z) = 0 \) for \( z \in U \), then \( f \equiv 0 \) on \( \Omega \).

**Proof.** Theorem 2.15 implies that the set \( D = \{ z \in \Omega : D^\alpha f(z) = 0 \text{ for all } \alpha \in \mathbb{N}^n \} \) is open. By continuity of \( D^\alpha f \), \( D \) is also closed. Since \( D \) is nonempty, connectedness implies \( D = \Omega \). \( \square \)

**Theorem 2.19** (Open mapping theorem). Let \( \Omega \subset \mathbb{C}^n \) be domain and suppose \( f \in \mathcal{O}(\Omega) \) is not constant. Then \( f(U) \) is open for any open set \( U \subset \Omega \).

**Proof.** It suffices to show that for any open ball \( B(a, r) \subset \Omega \), \( f(B(a, r)) \) is an open neighborhood of \( f(a) \). Theorem 2.18 implies that \( f|_{B(a, r)} \) is not constant, otherwise \( f \) would be constant on \( \Omega \). Choose \( p \in B(a, r) \) such that \( f(p) \neq f(a) \), and define \( h(\lambda) = f(a + \lambda(p-a)) \) for \( \lambda \in \Delta = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \). Then \( h \) is non constant on \( \Delta \) and holomorphic. By one variable open mapping theorem, \( h(\Delta) \subset f(B(a, r)) \) is a neighborhood of \( h(0) = f(a) \). \( \square \)

**Corollary 2.20** (The maximum modulus principle). Let \( \Omega \subset \mathbb{C}^n \) be an open set. Suppose \( f \in \mathcal{O}(\Omega) \) and that \( |f| \) has a local maximum at the point \( a \in \Omega \). Then \( f \) is constant on the connected component of \( \Omega \) containing \( a \).

**Proof.** ExerciseII. \( \square \)
3. Biholomorphic inequivalence of the ball and the polydisc

Definition 3.1 (Holomorphic map). If $\Omega \subset \mathbb{C}^n$ is a domain, then $F : \Omega \to \mathbb{C}^m$, where $F = (f_1, \ldots, f_m)$, is called a holomorphic map if each $f_j$ is a holomorphic function for $1 \leq j \leq m$.

The differential $dF(a)$ of a holomorphic map $F$ at $a \in \Omega$ is a complex linear map $\mathbb{C}^n \to \mathbb{C}^m$, with the complex matrix representation

$$F'(a) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1}(a) & \cdots & \frac{\partial f_1}{\partial z_n}(a) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1}(a) & \cdots & \frac{\partial f_m}{\partial z_n}(a) \end{bmatrix}$$

We call $F'(a)$ the derivative (or complex Jacobian matrix) of the holomorphic map $F$ at $a$.

Exercise II: Show that if $F : \Omega \to \mathbb{C}^n$ is a holomorphic map, where $\Omega \subset \mathbb{C}^n$ is a domain, then $\det J_R F(z) = |\det F'(z)|^2 \geq 0$, where $J_R F$ is the real $(2n \times 2n)$ Jacobian matrix of $F$.

Lemma 3.2 (Chain Rule). Let $D \subset \mathbb{C}^n$ and $\Omega \subset \mathbb{C}^m$ be domains. If $F = (f_1, \ldots, f_m) : D \to \Omega$ is holomorphic and $g \in \mathcal{O}(\Omega)$, then $g \circ F \in \mathcal{O}(D)$; moreover, for $a \in D$ and $1 \leq j \leq n$,

$$\frac{\partial (g \circ F)}{\partial z_j}(a) = \sum_{k=1}^m \frac{\partial g}{\partial w_k}(F(a)) \frac{\partial f_k}{\partial z_j}(a). \quad (3.3)$$

Theorem 3.4 (Inverse Mapping Theorem). Suppose $\Omega \subset \mathbb{C}^n$ is a domain and the holomorphic map $F : \Omega \to \mathbb{C}^n$ is non-singular at $a$, (i.e., $\det F'(a) \neq 0$). Then there are open neighborhoods $U$ of $a$ and $W$ of $b = F(a)$, such that $F|_U : U \to W$ is a homeomorphism with holomorphic inverse $H : W \to U$.

If $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ are domains, then we say the map $F : \Omega_1 \to \Omega_2$ is biholomorphic if $F$ is holomorphic homeomorphism with holomorphic inverse $F^{-1} : \Omega_2 \to \Omega_1$.

Two domains $\Omega_1$ and $\Omega_2$ are called biholomorphically equivalent or simply biholomorphic if there is a biholomorphism $F : \Omega_1 \to \Omega_2$.

The Riemann mapping theorem says that any simply connected domain in $\mathbb{C}$ (which is not whole of $\mathbb{C}$) is biholomorphic to the unit disc. H. Poincaré was the first to discover that the generalization fails in higher dimension:

Theorem 3.5. There exists no biholomorphic map

$$F : P(0,1) \to B(0,1)$$

between the polydisc and the unit ball in $\mathbb{C}^n$ if $n > 1$.

Proof. (Proof taken from [1])

For simplicity, we will assume $n = 2$. Let $\Delta = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$ be the open unit disc in $\mathbb{C}$.

Suppose $F = (f_1, f_2) : \Delta \times \Delta \to B = B(0,1) \subset \mathbb{C}^2$ be a biholomorphic map. For each fixed $w \in \Delta$, we define the holomorphic map $F_w : \Delta \to \mathbb{C}^2$ by

$$F_w(z) = \left( \frac{\partial f_1}{\partial w}(z,w), \frac{\partial f_2}{\partial w}(z,w) \right).$$

Let $\{z_\nu\} \subset \Delta$ be a sequence with $|z_\nu| \to 1$. We apply Montel’s theorem to the bounded sequence of holomorphic maps $\{F(z_\nu, \cdot)\}$ in the second variable to obtain a subsequence $\{z_{\nu_j}\}$, such that $\{F(z_{\nu_j}, \cdot)\}$ converges compactly in $\Delta$ to a holomorphic map $\varphi : \Delta \to \overline{B}$.

Since $F$ is biholomorphic, we must have $F(z_{\nu_j}, w) \to bB$ for every $w \in \Delta$ as $z_\nu \to b\Delta$. 


Hence $\varphi(\Delta) \subset bB$, i.e., if $\varphi = (\varphi_1, \varphi_2)$ then $|\varphi_1(w)|^2 + |\varphi_2(w)|^2 = 1$ for all $w \in \Delta$. Applying $\partial^2 / \partial \overline{w} \partial w$ to this equation, we get $|\varphi_1'(w)|^2 + |\varphi_2'(w)|^2 = 0$, so $\varphi' \equiv 0$ on $\Delta$. This gives

$$\lim_{z, w \to b \Delta} F_w(z) = 0.$$ 

This means $F_w$ extends continuously to $\overline{\Delta}$, with boundary values 0. This is a contradiction because by maximum modulus principle, $F_w \equiv 0$ on $\Delta$, which means $F(z, w)$ is independent of $w$, which means $F$ cannot be injective.

\[\square\]

References