Let $\Omega \subset \mathbb{C}^n$ be a domain. Recall from Section 1.4 of Lecture 1, that the $\bar{\partial}$ method consists of four steps, the third of which involves finding solutions to the equation

$$\bar{\partial}u = \alpha,$$  \hspace{1cm} (0.1)

where $\alpha \in \bar{\partial} \in \Lambda^{p,q}(\Omega)$ is a closed $(p,q)$-form with coefficients in $C^\infty(\Omega)$. The proof of existence of solutions to equation (0.1) requires tools from functional analysis which we develop now.

**Theorem 0.2** (Riesz representation theorem for Hilbert spaces). Let $\lambda$ be a bounded linear functional on a Hilbert space $H$. Then there exists a unique $g \in H$ such that

$$\lambda(f) = (f,g)$$  \hspace{1cm} for all $f \in H$.

Moreover, $\|g\| = \|\lambda\|$.

**Proof.** See [? , Theorem 5.3]. \hfill \square

1. Closed operators

An operator $T$ from a Hilbert space $H_1$ to another Hilbert space $H_2$ is a $\mathbb{C}$-linear map from a linear subspace $\text{Dom}(T)$ of $H_1$ into $H_2$. If $\text{Dom}(T)$ is dense in $H_1$, we say that $T$ is densely defined. We will assume that all our operators are densely defined. The subspace $\text{Dom}(T)$ is called the domain of $T$. Recall that an operator $T : H_1 \to H_2$ is said to be bounded if there is a $C > 0$ such that for all $f \in \text{Dom}(T)$ we have

$$\|Tf\|_{H_2} \leq C \|f\|_{H_1}.$$  

The analysis of unbounded operators becomes interesting if a densely defined unbounded operator is also closed.

**Definition 1.1.** A linear operator $T : H_1 \longrightarrow H_2$ is closed if its graph $\Gamma(T)$

$$\Gamma(T) = \{(f,Tf) \in H_1 \times H_2 | f \in \text{Dom}(T)\}$$

is a closed subspace of $H_1 \times H_2$, where $H_1 \times H_2$ is given the product topology, which is induced by the norm:

$$\|(h_1,h_2)\|_{H_1 \times H_2}^2 = \|h_1\|_{H_1}^2 + \|h_2\|_{H_2}^2.$$  

It is clear that this definition can be reformulated as below.

**Proposition 1.2.** The following are equivalent:

1. A linear operator $T : H_1 \longrightarrow H_2$ is closed.
2. For every sequence $(f_n)$ in $\text{Dom}(T)$ such that $f_n \to f$ in $H_1$ and $Tf_n \to g$ in $H_2$, we have $f \in \text{Dom}(T)$ and $g = Tf$.

Recall the following fundamental result from functional analysis.

**Theorem 1.3** (Closed Graph Theorem). A linear operator $T : H_1 \to H_2$ defined on all of $H_1$ is closed iff it is continuous.
**Definition 1.4** (Adjoint of an operator). Let $T : H_1 \rightarrow H_2$ be an operator. The adjoint $T^* : H_2 \rightarrow H_1$ of $T$ is the operator with

$$\text{Dom}(T^*) = \{ g \in H_2 | f \mapsto (Tf, g) \text{ is a bounded functional for all } f \in \text{Dom}(T) \}$$

and satisfying

$$(Tf, g)_{H_2} = (f, T^*g)_{H_1}$$

for all $f \in \text{Dom}(T)$ and $g \in \text{Dom}(T^*)$.

The existence of the adjoint follows from the Riesz representation theorem. Exercise.

**Exercise III**. Let $T : H_1 \rightarrow H_2$ be a densely defined operator. Show that $T^*$ is closed.

**Lemma 1.5**. Let $T : H_1 \rightarrow H_2$ be an operator. Then

1. $(\ker(T))^\perp = \text{Range}(T^*)$
2. $	ext{Range}(T) = (\ker(T^*))^\perp$

**Proof.** Exercise IV. □

We now turn to the class of unbounded operators that we are interested in.

**2. Differential operators**

Let $\Omega \subset \mathbb{R}^n$ be a domain. We equip the space $C_0^\infty(\Omega)$ with the following inner product

$$(f, g) = \int_\Omega f \overline{g} dV \quad \text{for all } f, g \in C_0^\infty(\Omega). \quad (2.1)$$

A differential operator $A : C_0^\infty(\Omega) \rightarrow C_0^\infty(\Omega)$ of order $m$ with coefficients in $C^\infty(\Omega)$ is given by its action on a function $u$ as

$$Au = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha u, \quad (2.2)$$

where $|\alpha| = \alpha_1 + \cdots + \alpha_n$. We assume that at least one function $a_\alpha$ with $|\alpha| = m$ does not vanish identically.

We want to think of these differential operators as operators acting on $L^2(\Omega)$ or perhaps some weighted $L^2$-space. Such a “Hilbert space realization” of a differential operator is never bounded, provided the order $m$ of the operator is at least 1. Instead of giving a general proof of this, let us illustrate why this is the case using an example. Let the operator $D : L^2(0, 1) \rightarrow L^2(0, 1)$ with $\text{Dom}(D) = C_0^\infty(0, 1)$ be given by

$$Df(x) = f'(x) \quad \text{for all } x \in (0, 1).$$

Then if $\phi \in C_0^\infty((0, 1))$ is a nonzero compactly supported smooth function, and we set $\phi_\epsilon(x) = \epsilon^{-\frac{1}{2}} \phi(\frac{x}{\epsilon})$, we have that $\|\phi_\epsilon\|$ is independent of $\epsilon$, but $\|D\phi_\epsilon\| \rightarrow \infty$ as $\epsilon \rightarrow 0$. This idea easily generalizes to arbitrary differential operators of order $\geq 1$.

**2.1. Formal adjoint of a differential operator.**

**Definition 2.3**. Given a differential operator $A : C_0^\infty(\Omega) \rightarrow C_0^\infty(\Omega)$, let $A' : C_0^\infty(\Omega) \rightarrow C_0^\infty(\Omega)$ be the operator satisfying

$$(Af, g) = (f, A'g) \quad \text{for all } f, g \in C_0^\infty(\Omega). \quad (2.4)$$

The operator $A'$ is said to be the formal adjoint of the differential operator $A$. 
We now show that the formal adjoint $A'$ of a differential operator $A$ of order $m$ exists and is a differential operator of order $m$. Let $A$ be as in (2.2). For functions $f, g \in C_0^\infty(\Omega)$ we obtain the following identity by integrating by parts
\[
\int_\Omega (a_\alpha D^\alpha f) \overline{g} d\nu = (-1)^{|\alpha|} \int_\Omega f \overline{D^\alpha (a_\alpha g)} d\nu, \tag{2.5}
\]
where $\alpha$ is a multi-index with $|\alpha| \leq m$. Let $A' : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ be the differential operator given by
\[
A'u = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha (a_\alpha(x))u \quad \text{for all } u \in C_0^\infty(\Omega),
\]
where the multi-indices $\alpha$ and the coefficients $a_\alpha$ are as in (2.2). In view of equation (2.5) we see that the operator $A'$ is the formal adjoint of the differential operator $A$ in (2.2).

Exercise V. Let $\Omega \subset \mathbb{R}^n$ be a domain with $C^1$ boundary. Let $(\nu_1, \ldots, \nu_n)$ be the outward pointing unit vector to $\partial \Omega$ and $d\sigma$ be the surface measure. Show that
\[
\int_\Omega u \frac{\partial v}{\partial x_j} d\nu = \int_{\partial \Omega} uv \nu_j d\sigma - \int_\Omega v \frac{\partial u}{\partial x_j} d\nu.
\]

### 2.2. Realizations of differential operators.

Let $\Omega$ be a domain in $\mathbb{R}^n$ and $A : C_0^\infty(\Omega) \to C_0^\infty(\Omega)$ be a differential operator. The operator $A$ gives rise to several closed operators on the Hilbert space $L^2(\Omega)$ two of which are given below.

**Definition 2.6** (Weak maximal realization). For $f, g \in L^2(\Omega)$ we say that $A f = g$ weakly if for all $\phi \in C_0^\infty(\Omega)$ we have
\[
(f, A' \phi) = (g, \phi).
\]
The weak maximal realization $A_{\max} : L^2(\Omega) \hookrightarrow L^2(\Omega)$ associated to $A$ has domain
\[
\text{Dom}(A_{\max}) = \{ f \in L^2(\Omega) | \exists g \in L^2(\Omega) \text{ such that } Af = g \text{ weakly} \}
\]
and $A_{\max} f = Af$ weakly.

**Definition 2.7** (Strong minimal realization). The strong minimal realization $A_{\min} : L^2(\Omega) \hookrightarrow L^2(\Omega)$ is the closure of the densely defined operator $A|_{C_0^\infty(\Omega)}$ on $L^2(\Omega)$ in the graph topology. That is, $\text{Dom}(A_{\min})$ consists of those $f \in L^2(\Omega)$ for which there is a $g \in L^2(\Omega)$ and a sequence $(f_n) \subset C_0^\infty(\Omega)$ such that $f_n \to f$ and $A f_n \to g$ in $L^2(\Omega)$.

More generally differential operators may act on sections of vector bundles. The $\overline{\partial}$ operator for example, acts on exterior powers of the cotangent bundle of a complex manifold. Then the operator $\overline{\partial}$ is locally given by a matrix of partial differential operators. Consider the following example.

Let $\Omega$ be a domain in $\mathbb{C}^n$. Then $\Lambda^{0,1}(\Omega) \cong (C^\infty(\Omega))^n$ and let $d\overline{z}_1, \ldots, d\overline{z}_n$ be the ordered basis for $\Lambda^{0,1}(\Omega)$. Then the operator $\overline{\partial} : C^\infty(\Omega) \to \Lambda^{0,1}(\Omega)$ is given by the matrix
\[
\overline{\partial} f = \left[ \frac{\partial}{\partial \overline{z}_1} \ldots \frac{\partial}{\partial \overline{z}_n} \right] f \quad \text{for all } f \in C^\infty(\Omega).
\]

**Lemma 2.8.** The operators $A_{\max} : L^2(\Omega) \hookrightarrow L^2(\Omega)$ and $A_{\min}' : L^2(\Omega) \hookrightarrow L^2(\Omega)$ are adjoints of each other.

**Proof.** See [?, Lemma 4.3].
3. DOMAINS OF HILBERT SPACE ADJOINTS

Let us do a computation to understand the domain of an adjoint. Consider the operator

\[ D = \left( \frac{d}{dx} \right)_{\text{max}} : L^2(0, 1) \rightarrow L^2(0, 1) \]

which is the maximal realization of the differential operator \( \frac{d}{dx} \) acting on functions on the interval \((0, 1)\). Assume that \( u, v \in C^1([0, 1]) \), and note that

\[
(Du, v)_{L^2} = \int_0^1 u'(x)v(x)dx = \int_0^1 u(x)(-v'(x))dx + (u(1)v(1) - u(0)v(0)).
\]

It is easy to see that the map \( u \mapsto u(1)v(1) - u(0)v(0) \) is bounded in the norm of \( L^2(0, 1) \) if and only if \( v(1) = v(0) = 0 \). On the other hand the map

\[
u \mapsto \int_0^1 u(x)(-v'(x))dx
\]

is bounded. Therefore we conclude that

\[ \text{Dom}(D^*) \cap C^1([0, 1]) = \{ v \in C^1([0, 1]) | v(1) = v(0) = 0 \}, \]

i.e., the domain of the adjoint is subject to certain boundary conditions. As we will see later, this fact plays a crucial role.

4. \( L^2 \) EXISTENCE THEOREM FOR \( \bar{\partial} \)

**Lemma 4.1.** Let \( T : H_1 \rightarrow H_2 \) be a linear, closed, densely defined operator. The following conditions on \( T \) are equivalent.

1. Range\((T)\) is closed.
2. There is a constant \( C > 0 \) such that \( \|f\|_{H_1} \leq C \|Tf\|_{H_2} \) for all \( f \in \text{Dom}(T) \cap \text{Range}(T^*) \).
3. Range\((T^*)\) is closed.
4. There is a constant \( C > 0 \) such that \( \|f\|_{H_2} \leq C \|T^*f\|_{H_1} \) for all \( f \in \text{Domain}(T^*) \cap \text{Range}(T) \).

The best constants in (2) and (4) are the same.

**Proof.** See [?], Lemma 4.1.1. \( \square \)

4.1. \( \bar{\partial} \) as a Hilbert space operator. Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \) and \( \varphi : \Omega \rightarrow \mathbb{R} \) be a strictly plurisubharmonic function. Let \( L^2_{0,1}(\Omega, e^{-\varphi}) \) be the Hilbert space

\[
L^2_{0,1}(\Omega, e^{-\varphi}) = \left\{ f = \sum_{j=1}^n f_j dz_j : \sum_{j=1}^n \int_{\Omega} |f_j|^2 e^{-\varphi} dV < \infty \right\}.
\]

The space \( L^2_{0,1}(\Omega, e^{-\varphi}) \) is equipped with the following inner product

\[
(f, g)_{\varphi} = \sum_{j=1}^n \int_{\Omega} f_j \overline{g_j} e^{-\varphi} dV \quad \text{for all } f, g \in L^2_{0,1}(\Omega, e^{-\varphi}).
\]

By abuse of notation, we denote the weak maximal realization of the differential operator \( \bar{\partial} : \Lambda(\Omega) \rightarrow \Lambda^{0,1}(\Omega) \) by \( \bar{\partial} : L^2(\Omega, e^{-\varphi}) \rightarrow L^2_{0,1}(\Omega, e^{-\varphi}) \), the latter being a Hilbert space operator. We denote by \( \bar{\partial}^*_\varphi : L^2_{0,1}(\Omega, e^{-\varphi}) \rightarrow L^2(\Omega, e^{-\varphi}) \) the Hilbert space adjoint of the operator \( \bar{\partial} \).
We now present the main application of functional analysis in the proof of Hörmander’s theorem.

**Theorem 4.2.** Let $\Omega \subset \mathbb{C}^n$ be a bounded domain, and let $\phi \in C^1(\overline{\Omega})$. That there is a constant $K > 0$ such that for every $(0,1)$-form $g \in \text{Dom}(\bar{\partial}) \cap \text{Dom}(\bar{\partial}_\phi^*)$ the following inequality holds

$$\|\bar{\partial}g\|_\phi^2 + \|\bar{\partial}_\phi^*g\|_\phi^2 \geq K^2 \|g\|_\phi^2 \quad (4.3)$$

Then for all $f \in L^2_{0,1}(\Omega)$ with $\bar{\partial}f = 0$, there is a $u \in L^2(\Omega)$ such that $\bar{\partial}u = f$

and

$$\|u\|_\phi \leq \frac{1}{K} \|g\|_\phi \quad (4.6)$$

**Remark.** It will be shown that inequality $(4.3)$ holds for a smoothly bounded pseudoconvex domain $\Omega$ is $\phi$ is a strictly psh function which is $C^2$ in $\Omega$, and $K$ is such that $i\partial\bar{\partial}\phi \geq K$. This is the basic estimate of $L^2$ theory who proof will take up most of the remaining week.

**Proof.** Since $\bar{\partial}^2 = 0$, we have $(\bar{\partial}_\phi^*)^2 = 0$ and hence

$$\text{Range}(\bar{\partial}) \subset \ker(\bar{\partial}) \quad (4.4)$$

and

$$\text{Range}(\bar{\partial}_\phi^*) \subset \ker(\bar{\partial}_\phi^*) \quad (4.5)$$

since $\ker(\bar{\partial})$ and $\ker(\bar{\partial}_\phi^*)$ are closed subspaces of $L^2(\Omega, e^{-\phi})$ and $L^2_{0,1}(\Omega, e^{-\phi})$ respectively. It follows from $(4.3)$ that

$$\|\bar{\partial}_\phi^*g\|_\phi^2 \geq K^2 \|g\|_\phi^2 \quad (4.6)$$

for all $(0,1)$-forms $g \in \text{Dom}(\bar{\partial}_\phi^*) \cap \ker(\bar{\partial})$.

By equation $(4.5)$, the inequality $(4.6)$ holds for all $g \in \text{Dom}(\bar{\partial}_\phi^*) \cap \text{Range}(\bar{\partial})$ and applying Lemma 4.1 to the densely defined, closed operator $\bar{\partial} : L^2(\Omega, e^{-\phi}) \rightarrow L^2_{0,1}(\Omega, e^{-\phi})$ we get that $\text{Range}(\bar{\partial}_\phi^*)$ is closed in $L^2(\Omega, e^{-\phi})$ and $\text{Range}(\bar{\partial})$ is closed in $L^2_{0,1}(\Omega, e^{-\phi})$.

Let $L : \text{Range}(\bar{\partial}_\phi^*) \rightarrow \mathbb{C}$ be the linear functional given by

$$L(\bar{\partial}_\phi^*g) = (g, f)_\phi \quad \text{for all } g \in \text{Dom}(\bar{\partial}_\phi^*).$$

We now show that $L$ is a bounded operator. Let $g \in \text{Dom}(\bar{\partial}_\phi^*)$ be given. Then $g$ can be written as $g = g_1 + g_2$ where $g_1 \in \text{Dom}(\bar{\partial}_\phi^*) \cap \ker(\bar{\partial})$ and $g_2 \in \text{Dom}(\bar{\partial}_\phi^*) \cap (\ker(\bar{\partial}))^\perp$. Then we have

$$\|(f, g_1)_\phi\| = \|f\|_\phi \|g_1\|_\phi \leq \frac{1}{K} \|f\|_\phi \|\bar{\partial}_\phi^*g_1\|_\phi.$$  

Also, $(f, g_2)_\phi = 0$, since $f \in \ker(\bar{\partial})$ and $g_2 \in (\ker(\bar{\partial}))^\perp$. Also $\bar{\partial}_\phi^*g_2 = 0$ since $(\ker(\bar{\partial}))^\perp = \text{Range}(\bar{\partial}_\phi^*)$. Thus, we have

$$\|(f, g)_\phi\| = \|(f, g_1)_\phi\| \leq \frac{1}{K} \|f\|_\phi \|\bar{\partial}_\phi^*g_1\|_\phi.$$  

This shows that $\|L\| \leq \frac{1}{K} \cdot \|f\|_\phi$ and hence $L$ is bounded. By Hahn-Banach theorem, $L$ extends to a bounded linear functional $\tilde{L}$ defined on all of $L^2(\Omega, e^{-\phi})$ such that $\|\tilde{L}\| = \|L\|.$
Then, by the Riesz Representation theorem, there is a $u \in L^2(\Omega, e^{-\varphi})$ such that for all $g \in \text{Dom}(\partial^*_\varphi)$ we have

$$(\partial^*_\varphi g, f)_{\varphi} = L(\partial^*_\varphi g) = \tilde{L}(\partial^*_\varphi g) = (\partial^*_\varphi g, u)_{\varphi} \quad (4.7)$$

and

$$\|u\|_\varphi = \|\tilde{L}\| = \|L\| \leq \frac{1}{K} \|f\|_\varphi. \quad (4.8)$$

It follows from equation (4.7) that $\bar{\partial} u = f$ weakly. \qed