5. Pseudoconvexity

Domains with regular (smooth) boundary are studied first. This has a great computational advantage, as well as a conceptional one. Results on domains with less boundary regularity can often be deduced from their smooth approximants.

5.1. Defining functions. Let \( \mathbb{R}^N \) denote \( N \)-dimensional real euclidean space and \((a_1, \ldots, a_N)\) denote the standard coordinates on \( \mathbb{R}^N \). As with subharmonic functions, several notions are most naturally stated in \( \mathbb{R}^N \).

**Definition 5.1.** Let \( \Omega \subset \mathbb{R}^N \) be a domain. A real-valued function \( r \in C^1(\bar{\Omega}) \) is a defining function for \( \Omega \) if

\[
\Omega = \{ a \in \mathbb{R}^N : r(a) < 0 \} \quad \text{and} \quad \text{dr} \neq 0 \text{ on } \{ a \in \mathbb{R}^N : r(a) = 0 \} =: b\Omega.
\]

Higher levels of regularity for \( b\Omega \) are prescribed by requiring \( r \) to belong to smoother functions spaces, e.g. \( C^k(\bar{\Omega}) \), \( C^\infty(\bar{\Omega}) \). When \( r \) can be chosen in \( C^\infty(\bar{\Omega}) \), \( b\Omega \) is called smooth and \( \Omega \) is called a smoothly bounded domain.

Defining functions are not unique. However if \( r \) is a fixed defining function and \( \rho \) is any other defining function for \( \Omega \), there exists \( h \in C^1(\bar{\Omega}) \) with \( h > 0 \) in a neighborhood of \( b\Omega \) such that \( \rho = h \cdot r \). Exercise II. When \( b\Omega \) has higher than \( C^1 \) regularity, \( h \) is in the corresponding function space.

Another common definition for \( \Omega \) to have \( C^k \) boundary is to require that \( b\Omega \) be a \( C^k \) manifold. If \( \Omega \) has \( C^k \) boundary in this sense, it is not difficult to construct a \( C^k \) defining function – in fact, infinitely many – in the sense of Definition 5.1.

One useful choice of defining function involves the euclidean structure of \( \mathbb{R}^N \). If \( F \subset \mathbb{R}^N \) is closed, let \( d_F(a) = \inf \{|a - f| : f \in F\} \). The signed distance function to \( b\Omega \) is defined

\[
\delta_{b\Omega}(a) = \begin{cases} 
-d_{b\Omega}(a), & a \in \Omega \\
 d_{b\Omega}(a), & a \in \Omega^c
\end{cases}
\]

If \( b\Omega \) is a \( C^k \) manifold, \( \delta_{b\Omega} \) is a \( C^k \) defining function. Exercise III.

5.2. Tangent spaces to \( b\Omega \).

**Definition 5.2.** Let \( \Omega \subset \mathbb{R}^N \) be a domain with \( C^1 \) boundary, \( p \in b\Omega \), and \( r \) a defining function for \( \Omega \).

The (real) tangent space to \( b\Omega \) at \( p \) is

\[
T_p(b\Omega) = \left\{ u = (u_1, \ldots, u_N) \in \mathbb{R}^N : \sum_{j=1}^N \frac{\partial r}{\partial a_j}(p) u_j = 0 \right\}.
\]
Note that $T_p(b\Omega)$ is a vector space, has dimension $N - 1$, and does not depend on the choice of defining function $r$. **Exercise I.**

Recall the standard coordinates on $\mathbb{C}^n$ are being denoted $z_j = x_j + iy_j$, $j = 1, \ldots, n$. If $\Omega$ is a domain in $\mathbb{C}^n$ (in particular $N = 2n$ above), writing Definition 5.2 in terms of $z_j$ derivatives is both useful and enlightening. First, if the underlying structure on $\mathbb{R}^{2n}$ is given with the order $(x_1, y_1, \ldots, x_n, y_n)$, then a vector $(u_1, v_1, \ldots, u_n, v_n) \in \mathbb{R}^{2n}$ belongs to $T_p(b\Omega)$ if

$$
\sum_{j=1}^n \frac{\partial r}{\partial x_j}(p)u_j + \sum_{j=1}^n \frac{\partial r}{\partial y_j}(p)v_j = 0. \quad (5.3)
$$

Let $w_j = u_j + iv_j$ to express vectors using the structure of $\mathbb{C}^n$. It follows that

$$
\text{(5.3)} \iff \text{Re} \left( \sum_{j=1}^n \frac{\partial r}{\partial z_j}(p)w_j \right) = 0 \quad \text{**Exercise I.}.
$$

Continue letting $\Omega \subset \mathbb{C}^n$ and use complex notation. Note that $w = (w_1, \ldots, w_n) \in T_p(b\Omega)$ does not necessarily force $iw \in T_p(b\Omega)$, i.e., $T_p(b\Omega)$ is not a complex vector space. A distinguished subspace of $T_p(b\Omega)$ plays a large role in the sequel.

**Definition 5.4.** Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^1$ boundary, $p \in b\Omega$, and $r$ a defining function for $\Omega$. The **complex tangent space to $b\Omega$ at $p$** is

$$
T^c_p(b\Omega) = \left\{ w \in \mathbb{C}^n : \sum_{j=1}^n \frac{\partial r}{\partial z_j}(p)w_j = 0 \right\}.
$$

Elementary arguments show $T^c_p(b\Omega) \subset T_p(b\Omega)$ is the maximal complex subspace and $\dim_{\mathbb{R}} T_p^c(b\Omega) = 2n - 2$. **Exercise I.**

The single “direction” (from the vector space point of view) $X \in T_p(b\Omega) \setminus T^c_p(b\Omega)$ is often called the **bad tangential direction** to $b\Omega$ at $p$. The definite article is a little misleading: a generic tangential vector to $b\Omega$ has a non-zero component in the bad direction.

### 5.3. Convex domains.

**Definition 5.5.** Let $U \subset \mathbb{R}^N$ be open and $f \in C^2(U)$. The **Hessian of $f$ at $x \in U$** is the $N \times N$ matrix of functions

$$
H_f = \left( \frac{\partial^2 f}{\partial a_j \partial a_k} \right)_{j,k=1}^N.
$$

For vectors $v, w \in \mathbb{R}^N$, the Hessian acts on the pair $(v, w)$ by the prescription

$$
H_f[v, w] =: \sum_{j,k=1}^N \frac{\partial^2 f}{\partial a_j \partial a_k} v_j w_k. \quad (5.6)
$$

Evaluation at $x \in U$ is denoted $H_f[v, w](x)$.

A function $f \in C^2(U)$ is convex near $x \in U$ — in the usual sense that $f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b)$ for all $a, b$ close to $x$ and for all $\lambda \in [0, 1]$ — if and only if $H_f[v, w](y) \geq 0$ for all $y$ close to $x$ and for all $v \in \mathbb{R}^N \setminus \{0\}$. A similar condition turns out to characterize convex sets with sufficiently smooth boundaries in $\mathbb{R}^N$.

**Definition 5.7.** Let $\Omega \subset \mathbb{R}^N$ be a domain with $C^2$ boundary and $r$ be a defining function for $\Omega$. Say that $\Omega$ is convex at $p \in b\Omega$ if

$$
H_f[v, v](p) \geq 0 \quad \text{for all } v \in T_p(b\Omega).\quad (5.8)
$$

Say that $\Omega$ is strongly convex at $p$ if inequality (5.8) is strict for $v \neq 0$. 
The usual definition for $\Omega \subset \mathbb{R}^N$ to be convex is geometric: if $x, y \in \Omega$, then $\lambda x + (1-\lambda)y \in \Omega$ for all $\lambda \in [0,1]$. It is not obvious Definition 5.7 coincides with this definition, but this indeed holds. ExerciseII.

Inequality \((5.8)\) does not depend on the choice of defining function. ExerciseI. Also note that semi-definiteness in \((5.8)\) is only being prescribed in directions $v \in T_p(b\Omega)$. The function $r(x, y) = y - y^2 + x^2$ near $(0,0)$ is $\mathbb{R}^2$ locally defines a convex domain, but shows the Hessian of a general defining function may be negative in non-tangential directions. It is however possible to choose a defining function for any convex domain which is fully convex, see [?].

5.4. Pseudoconvexity. Now let $\Omega \subset \mathbb{C}^n$ with $C^2$ boundary. For complex analysis, the principal convexity notion on $\Omega$ has a parallel formulation to Definition 5.7.

Definition 5.9. Let $\Omega \subset \mathbb{C}^n$ be a domain with $C^2$ boundary and $r$ be a defining function for $\Omega$. Say that $\Omega$ is Levi pseudoconvex at $p \in b\Omega$ if
\[
\mathcal{H}_r[\xi, \bar{\xi}](p) \geq 0 \quad \text{for all } \xi \in T^C_p(b\Omega).
\]
Say that $\Omega$ is strongly pseudoconvex at $p$ if inequality \((5.10)\) is strict for $\xi \neq 0$.

Re-expressing Levi pseudoconvexity using notation (3.18) of Lecture 3, $\Omega$ satisfies Definition 5.9 if
\[
i\bar{\partial}\partial r[\xi, \bar{\xi}](p) \geq 0 \quad \text{for all } \xi \in T^C_p(b\Omega) \text{ and for all } p \in b\Omega,
\]
for any defining function $r$ for $\Omega$. Condition \((5.11)\) is independent of $r$ ExerciseI.

Some basic facts about pseudoconvexity are given in the next three Propositions.

Proposition 5.12. Any smoothly bounded convex domain $\Omega \subset \mathbb{C}^n$ is Levi pseudoconvex.

Proof. ExerciseII (Sketch) Use Taylor’s theorem. First show that
\[
\frac{1}{2} H_r[v, v] = \mathcal{H}_r[\xi, \xi] + \text{Re } \mathcal{Q}_r[\xi, \xi]
\]
where $\xi_j = v_{2j-1} + iv_{2j}$ and
\[
\mathcal{Q}_r[\xi, \xi] = \sum_{j,k=1}^n \frac{\partial^2 r}{\partial z_j \partial \bar{z}_k} \xi_j \xi_k.
\]
Apply this to $\xi$ and $i\xi$, if $\xi \in T^C_p(b\Omega)$.

Converse of this proposition is not true as, e.g., any smoothly bounded non-convex $\Omega \subset \mathbb{C}$ shows.

Proposition 5.13. Suppose $\Omega \subset \mathbb{C}^n$ with $C^2$ boundary, $p \in b\Omega$, $U$ is a neighborhood of $p$, and $w = F(z)$ is a biholomorphic map. Let $\tilde{\Omega} = F(U \cap \Omega)$.

Then
\[
\tilde{\Omega} \text{ is Levi pseudoconvex at } q = F(p) \iff \Omega \text{ is Levi pseudoconvex at } p.
\]

Proof. (Sketch) Check $\rho =: r \circ F^{-1}$ is a $C^2$ defining function for $\tilde{\Omega}$ near $q$. Show
\[
\partial r[\xi](p) = \partial \rho \left[ F'(p)\xi \right] \tag{5.12}
\]
Then show $i\bar{\partial}\partial r[\xi, \bar{\xi}](p) = i\bar{\partial}\partial \rho \left[ F'(p)\xi, F'(p)\bar{\xi} \right](q)$.

Proposition 5.14. Suppose $\Omega \subset \mathbb{C}^n$ with $C^2$ boundary and is Levi pseudoconvex near $p \in b\Omega$.

Then there exists a defining function $\rho$ on a neighborhood $U$ of $p$ such that for all $q \in U$
\[
i\bar{\partial}\partial \rho[\xi, \bar{\xi}](q) \geq 0 \quad \text{for all } \partial \rho[\xi](q) = 0.
\]
The level set $b\Omega^q = \{ z \in U : r(z) = r(q) \}$ is a hypersurface approximating $b\Omega$ near $p$. The condition $\partial\rho[\xi](q) = 0$ says that $\xi \in T^\circ_q (b\Omega^q)$; thus Proposition 5.14 says these level sets locally bound pseudoconvex domains.

Proposition 5.14 only holds locally. The Diederich-Fornaess worm is a counterexample to the global statement (from the outside).

5.4.1. Strongly pseudoconvex domains. Strong pseudoconvexity is much more malleable than pseudoconvexity. The next three results are fundamental.

Proposition 5.15. Let $\Omega \subset \mathbb{C}^n$ with $C^2$ boundary be strongly pseudoconvex near $p \in b\Omega$.

Then there exists a neighborhood $U$ of $p$, a defining function $\rho$ for $\Omega \cap U$, and a constant $c > 0$ such that

$$i\partial\bar{\partial}\rho[\xi, \bar{\xi}](q) \geq c|\xi|^2 \quad \text{for all } \xi \in \mathbb{C}^n.$$ 

The gain in Proposition 5.15 is that the form $i\partial\bar{\partial}\rho$ is positive definite in all directions $\xi$ and for all $q \in U$, not only for $\xi \in T^\circ_p (b\Omega)$ and $p \in b\Omega$.

Remark 5.16. If every $p \in b\Omega$ is strongly pseudoconvex, the local defining functions in 5.15 can be patched together: there is a neighborhood $\mathcal{C}$ of $b\Omega$ and a defining function $\rho$ on $\Omega \cap \mathcal{C}$ such that $i\partial\bar{\partial}\rho[\xi, \bar{\xi}](z) \geq c|\xi|^2$ for all $\xi \in \mathbb{C}^n$ and $z \in \mathcal{C}$. ExerciseI.

Proof. To be given. □

Proposition 5.17. Let $\Omega \subset \mathbb{C}^n$ with $C^2$ boundary be strongly pseudoconvex near $p \in b\Omega$.

Then there exists holomorphic coordinates $w = (w_1, \ldots, w_n)$ in a neighborhood $U$ of $p$ such that $\Omega \cap U$ is convex in the $w$ coordinates.

Proof. To be given. □

Proposition 5.17 is local. In general, a strongly pseudoconvex domain is not biholomorphically equivalent to a convex domain. Give example of torus??

The last basic result on strongly pseudoconvex domains involves the notion of an exhaustion function. If $D \subset \mathbb{C}^n$ is an open set (no boundary regularity needed), a function $g : D \to \mathbb{R}$ is an exhaustion function if for every $\mu \in \mathbb{R}$, the set $D_\mu = \{ z \in D : g(z) < \mu \}$ is relatively compact in $D$. An exhaustion function necessarily satisfies $g(z) \to \infty$ as $z \to bD$, and this condition is also sufficient if $D$ is bounded.

Proposition 5.18. If $\Omega \subset \subset \mathbb{C}^n$ is Levi strongly pseudoconvex, then $\Omega$ admits a strongly plurisubharmonic exhaustion function.

Proof. (Sketch) From Remark 5.16, choose a neighborhood $\mathcal{C}$ of $b\Omega$ and a strictly plurisubharmonic $r$ on $\Omega \cap \mathcal{C}$ such that $\Omega \cap \mathcal{C} = \{ z \in \mathcal{C} : r(z) < 0 \}$ and $dr(z) \neq 0$ for $z \in \mathcal{C}$.

WLOG, we can assume $r \in C^2(\overline{\Omega})$ (why?). Subtracting a large positive constant from $r$, can also assume $r < 0$ on all of $\Omega$ – note use of $\Omega \subset \subset \mathbb{C}^n$.

Direct computation shows $\phi = : -\log(-r)$ is strictly plurisubharmonic of $\Omega \cap \mathcal{C}$ ExerciseI. Let

$$\mu = : \inf \{ \mathcal{H}_\phi[\xi, \bar{\xi}](z) : z \in \Omega \setminus \mathcal{C} \text{ and } |\xi| = 1 \}.$$ 

Then $\mu > -\infty$. The function $\psi = : \phi + (\mu + 1)|z|^2$ works. □

Proposition 5.18 implies that strongly pseudoconvex domains $\Omega$ can be approximated from the inside and the outside by strongly pseudoconvex domains. This is a fundamental geometric fact about strongly pseudoconvex domains which does not hold on more general pseudoconvex domains. To record the state of affairs, first
Definition 5.19. A collection \( \{U_\alpha\} \subset \mathbb{C}^n \) of open sets is a neighborhood basis of some set \( D \subset \mathbb{C}^n \) if \( \cap_\alpha U_\alpha = D \).

If \( \Omega \subset \mathbb{C}^n \) is a domain, the Nebenhülle of \( \Omega \) is defined
\[
\mathcal{N}(\Omega) = \text{interior} \left\{ \cap U_\alpha : U_\alpha \text{ is pseudoconvex and } \Omega \subset U_\alpha \right\}.
\]
If \( \mathcal{N}(\Omega) \setminus \Omega \) has interior points, then \( \Omega \) is said to have nontrivial Nebenhülle.

Sard’s theorem and Proposition 5.18 imply

**Proposition 5.20.** Let \( \Omega \subset \mathbb{C}^n \) be a bounded, strongly pseudoconvex domain. There exist montone sequences of strongly pseudoconvex domains \( \{I_j\} \subset \Omega \) and \( \{O_j\} \supset \Omega \),
\[
I_j \subset I_{j+1} \quad \text{and} \quad O_j \supset O_{j+1} \quad \text{for all} \quad j,
\]
such that
\[
\text{(i) } \Omega = \cup I_j \quad \text{and} \quad \text{(ii) } O_j \text{ is a neighborhood basis for } \Omega.
\]
In particular, \( \mathcal{N}(\Omega) = \emptyset \).

**Proof.** To be discussed. \( \square \)

5.4.2. **Weakly pseudoconvex domains.** Consider the propositions in the previous section on weakly pseudoconvex domains (still with smooth boundary). The first two observations are negative results.

**Proposition 5.21** (Non-Proposition 5.15). There exist smoothly bounded pseudoconvex \( \Omega \subset \mathbb{C}^n \), \( p \in b\Omega \), and \( U \) a neighborhood of \( p \), for which every local defining function \( r \) for \( U \cap \Omega \) has the property that \( i\partial \bar{\partial} r(\nu, \nu)(q) < 0 \) if \( \nu \notin T^C_q(b\Omega) \) and \( q \in U \).

Give reference to Fornaess example.

**Proposition 5.22** (Non-Proposition 5.17). There exist smoothly bounded pseudoconvex \( \Omega \subset \mathbb{C}^n \) and \( p \in b\Omega \) such that for no neighborhood \( U \) does there exist a holomorphic coordinate system on \( U \) in which \( \Omega \cap U \) is convex.

Originally, Proposition 5.22 was discovered by Kohn-Nirenberg in the early 1970s. Simplifications are due to Fornæss. The Kohn-Nirenberg proof is discussed in Problem sets. For so-called “type 4” boundary points, issues remain unresolved.

A positive result, however, holds relative to 5.18.

**Proposition 5.23.** If \( \Omega \subset \subset \mathbb{C}^n \) is a Levi pseudoconvex domain with \( C^3 \) boundary, then \( \Omega \) admits a strongly plurisubharmonic exhaustion function.

**Proof.** To be given. \( \square \)

This result yields half of Proposition 5.20.

**Proposition 5.24.** If \( \Omega \subset \subset \mathbb{C}^n \) is a Levi pseudoconvex domain with \( C^3 \) boundary, then there exists an increasing sequence of strongly pseudoconvex domains \( \{\Omega_j\} \), \( \Omega_j \subset \Omega \), such that \( \Omega = \cup_{j=1}^{\infty} \Omega_j \).

Both Propositions 5.23 and 5.24 hold if \( b\Omega \) is only \( C^2 \), but the proofs are more difficult.

However the other half of Proposition 5.20 – approximating \( \Omega \) by pseudoconvex domains from the outside – does not hold on a general weakly pseudoconvex domain \( \Omega \). This was originally discovered by Diederich-Fornaess. Their domains \( W \) are known as “worm domains” and will be discussed in Problem sets, as well as the non-smoothly bounded Hartogs triangles which underlie the Diederich-Fornaess construction.