MSRI2018, LECTURE 7

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READ WITH CAUTION, TYPOS ABOUND.

7. L^2 estimates on $\bar\partial$: Part I

Throughout $\Omega \subset \mathbb{C}^n$ is a smoothly bounded domain (usually $b\Omega \in C^2$ suffices). The goal of this lecture and the next is to unpack the basic L^2 estimates that hold on $\bar\partial$. This is done in 3 passes: (i) for the usual L^2 structure on $\mathbb{C}^n$, (ii) for weighted L^2 norms – i.e., the celebrated Hörmander estimates, and (iii) for weighted L^2 norms on a twisted version of the $\bar\partial$-complex. The twisted-weighted estimates give results not accessible by Hörmander estimates – this will be clearer in applications discussed afterwards.

7.1. **Estimates in ordinary L^2.** The $\bar\partial$ operator in Section 3.1 is not *a priori* a Hilbert space operator. However it can be extended to such an operator, given its initial definition on $\Lambda^{p,q}(\Omega)$.

7.1.1. *Norms of forms.* First need *pointwise* inner products and norms on forms. If $\beta, \gamma \in \Lambda^{p,q}(\Omega)$ is expressed in the form (3.1), i.e.,

$$\beta = \sum_{\substack{|I|=p\\|J|=q}}{}' \beta_{IJ}\, dz^I \wedge d\bar z^J \quad \text{and} \quad \gamma = \sum_{\substack{|K|=p\\|L|=q}}{}' \gamma_{KL}\, dz^K \wedge d\bar z^L,$$

define

$$\langle \beta, \gamma \rangle = \sum_{\substack{|I|=p\\|J|=q}} \beta_{IJ}\bar\gamma_{IJ}. \tag{7.1}$$

Forms of different bi-degrees are simply declared to be pointwise perpendicular, i.e. if $\mu \in \Lambda^{p,q}(\Omega)$, $\nu \in \Lambda^{r,s}(\Omega)$ and $(p,q) \neq (r,s)$, then $\langle \mu, \nu \rangle = 0$. More generally, notice that $\langle \beta, \gamma \rangle$ in (7.1) only involves matching indices of β and γ.

A *global* inner product is defined by integration:

$$(\beta, \gamma) = \int_\Omega \langle \beta, \gamma \rangle \, dV, \tag{7.2}$$

where dV is the euclidean volume form. The associated L^2-norm is denoted $\|\beta\| = (\beta, \beta)^{\frac{1}{2}}$.

$\Lambda^{p,q}(\Omega)$ is not complete in $\|\cdot\|$. Let $L^2_{p,q}(\Omega)$ denote the completion of $\Lambda^{p,q}(\Omega)$ with respect to $\|\cdot\|$. A standard measure theory argument gives

$$L^2_{p,q}(\Omega) = \{(p,q) \text{ forms } \eta \text{ with measurable components} : \|\eta\| < \infty\}.$$

7.1.2. *Extending $\bar\partial$.* To turn $\bar\partial$ into an L^2 operator, define

$$D\left(\bar\partial\right) = \left\{u \in L^2_{p,q}(\Omega) : \exists \{u_j\} \in \Lambda_{p,q}\left(\overline{\Omega}\right) \text{ such that } \lim u_j = u \text{ and } \left\{\bar\partial u_j\right\} \text{ is Cauchy in } L^2_{p,q+1}(\Omega)\right\} \tag{7.3}$$

$D\left(\bar\partial\right)$ is the **domain of** $\bar\partial$. This set of forms can also be viewed as

$$D(\bar\partial) = \left\{u \in L^2_{p,q}(\Omega) : \bar\partial u \in L^2_{p,q+1}(\Omega)\right\},$$

where $\bar\partial$ acts on u as a distributional derivative. ExerciseII.

For $u \in D\left(\bar\partial\right)$, set

$$\bar\partial u = \lim \bar\partial u_j.$$

We point out an abuse of notation here. $\bar\partial u_j$ appearing on the RHS is well-defined since u_j is smooth, but the equation *defines* the meaning of $\bar\partial u$ for $u \in D(\bar\partial)$. The symbol $\bar\partial$, which denoted a pointwise-defined operator is used here differently. Notational abuse like this is wide-spread; it is also confusing (at first)! In any case, verify the definition given of $\bar\partial u$ is independent of $\{u_j\}$ ExerciseI.. Thus the equation defines an operator $\bar\partial : D(\bar\partial) \to L^2_{(\cdot,\cdot)}(\Omega)$, which coincides with the earlier definition of $\bar\partial v$ when $v \in \Lambda^{p,q}(\Omega)$.

The pair $\left(\bar\partial, D(\bar\partial)\right)$ is the maximal extension of $\bar\partial$, the operator $\bar\partial$ is densely defined on the L^2 spaces, and $\bar\partial$ is a closed operator. But $\bar\partial$ is *not* a bounded operator on L^2. Give example, here or in Lecture 6. Lecture 6 discusses all this in more detail.

From now on, $\bar\partial$ denotes the extended operator above.

7.1.3. *$\bar\partial^*$ and its domain.* Since $\bar\partial$ is closed and densely defined, it has a well-defined adjoint operator, $\bar\partial^*$. The adjoint operator is also unbounded, but is a closed operator on the domain given by

$$D\left(\bar\partial^*\right) = \left\{v \in L^2_{p,q}(\Omega) : \exists C > 0 \text{ such that } \left|\left(\bar\partial u, v\right)\right| \le C\|u\| \text{ for all } u \in D\left(\bar\partial\right)\right\} \tag{7.4}$$

To define $\bar\partial^*$ on $D\left(\bar\partial^*\right)$, note (7.4) says $u \to \left(\bar\partial u, v\right)$ is a bounded linear functional on $D(\bar\partial)$. This functional extends to entire L^2 space since $D(\bar\partial)$ is dense. The Riesz representative of this functional is, by definition, $\bar\partial^* v$.

Thus the usual adjoint relationship holds

$$\left(u, \bar\partial^* v\right) = \left(\bar\partial u, v\right) \qquad \forall u \in D\left(\bar\partial\right), v \in D\left(\bar\partial^*\right). \tag{7.5}$$

To do serious computations, two elements are needed: (i) an expression of $\bar\partial^*$ as a differential operator, and (ii) a (dense) subspace of $D(\bar\partial^*)$ where calculus can be applied.

7.1.4. *On integration by parts.* To belong to $D(\bar\partial^*)$, a form must satisfy conditions "on" $b\Omega$: terms arising during integration by parts must vanish in a suitable sense. For a general form, these conditions are abstract and difficult to characterize. But for forms that are smooth up to $b\Omega$, the boundary conditions become concrete and relatively simple.

Let r be a defining function for Ω. Recall the divergence theorem: if $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain, $F = (F_1, \ldots, F_N)$ is a smooth vector field, and ν is the outward unit normal field to $b\Omega$, then

$$\int_\Omega \nabla \cdot F \, dV = \int_{b\Omega} F \cdot \nu \, dS,$$

where dS is surface measure on $b\Omega$. Let $(x_1, \ldots, x_{2n})$ be the underlying real coordinates on $\mathbb{C}^n$ in any fixed order. If $X = \sum c_i \frac{\partial}{\partial x_i}$ is a first order differential operator with coefficients

$c_i \in C^\infty(\overline{\Omega})$ and $a, b \in C^\infty(\overline{\Omega})$, it follows

$$\begin{aligned}(Xa, b) &= \int_\Omega \sum_{i=1}^{2n} c_i \frac{\partial a}{\partial x_i} \bar{b}\, dV \\ &= \sum_{i=1}^{2n} \int_\Omega \frac{\partial}{\partial x_i}\left(c_i a \bar{b}\right)\, dV - \int_\Omega a \frac{\partial}{\partial x_i}\left(c_i \bar{b}\right)\, dV \\ &= \int_\Omega \nabla \cdot \left(c_1 a\bar{b}, \ldots, c_{2n} a\bar{b}\right)\, dV + \left(a, X'b\right) \\ &= (a, X'b) + \int_{b\Omega} \sum_{i=1}^{2n} c_i \frac{\partial r}{\partial x_i} a\bar{b} \frac{dS}{|\nabla r|}. \end{aligned} \tag{7.6}$$

The last equality defines the *formal adjoint* X' of X.

The same considerations apply to forms, and to form-valued operators on forms (like $\bar{\partial}$). This is true because we have extended the L^2 inner product on functions "by linearity" to our spaces of forms.

A specialized defining functions simplifies (7.6), which is occasionally useful. Say a defining function ρ for a smoothly bounded domain Ω is *normalized* if $\|\partial\rho\| = 1$ on $b\Omega$. Can always find a normalized defining function Exercise0.. For the Cauchy-Riemann operators IBP can be written: if ρ is a normalized defining function for Ω

$$\left(\frac{\partial f}{\partial z_k}, g\right) = -\left(f, \frac{\partial g}{\partial \bar{z}_k}\right) + \int_{b\Omega} f\bar{g} \frac{\partial \rho}{\partial z_k}\, dS, \tag{7.7}$$

for $f, g \in \Lambda^{0,0}(\Omega)$. Similar expressions hold for forms.

7.1.5. *The space $\mathcal{D}^{p,q}$.* Now consider smooth up-to-the-boundary forms in $D\left(\bar{\partial}^*\right)$. Let

$$\mathcal{D}^{p,q} = D\left(\bar{\partial}^*\right) \cap \Lambda^{p,q}(\overline{\Omega}).$$

It is easy to show that

Proposition 7.8. *A form $u = \sum u_{IM} dz^I \wedge d\bar{z}^M \in \mathcal{D}^{p,q}$ if and only if*

$$\sum_{k=1}^{n} u_{I,kJ}(z) \frac{\partial r}{\partial z_k}(z) = 0 \qquad \text{for all } |I| = p, |J| = q-1, \text{ and } z \in b\Omega. \tag{7.9}$$

Proof. Focus on the boundary integral(s) arising from (7.7). ExerciseII. □

Specialize to $(0,1)$-forms. Then

$$\mathcal{D}^{0,1} = \left\{ v \in \Lambda^{0,1}\left(\overline{\Omega}\right) : \sum_{k=1}^{n} \frac{\partial r}{\partial z_k} v_k = 0 \text{ on } b\Omega \right\}.$$

Moreover, **if** $v = \sum_{k=1}^n v_k\, d\bar{z}_k \in \mathcal{D}^{0,1}$, (7.7) and (7.5) imply

$$\bar{\partial}^* v = -\sum_{k=1}^{n} \frac{\partial v_k}{\partial z_k}. \tag{7.10}$$

ExerciseI.The operator on the RHS of (7.10) is the formal adjoint of $\bar{\partial}$ and denoted ϑv. ϑv is defined (as a differential operator) for any smooth v; the computation just given shows $\bar{\partial}^* v = \vartheta v$ **when** $v \in \mathcal{D}^{0,1}$.

This is useful because

Proposition 7.11. *$\mathcal{D}^{p,q}$ is dense in $D\left(\bar\partial^*\right)\cap D(\bar\partial)$ in the norm*

$$\|u\|_{\mathcal{G}} =: \|u\| + \left\|\bar\partial u\right\| + \left\|\bar\partial^* u\right\|.$$

Proof. The proof is a bit intricate. It hinges on the basic Friedrich mollifier technique, but the tangential and "normal" components of a form u must be handled separately. See [**?**] pages 94–98 or [**?**] pages 70–74 of a detailed proof. □

Remark 7.12. $\Lambda_0^{p,q}(\Omega$ is **not** dense in $D\left(\bar\partial^*\right)\cap D(\bar\partial)$ in the graph norm $\|\cdot\|_{\mathcal{G}}$.

The main issue can be seen with ordinary Sobolev spaces. For $\Omega\subset\mathbb{R}^N$, $k\in\mathbb{Z}^+$, and $f\in C^k(\Omega)$, define the norm

$$\|f\|_k^2 = \sum_{\|\alpha|\le k}\|D^\alpha f\|^2,$$

where $\|\cdot\|$ is the L^2-norm and $D^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_N}}{\partial x_N^{\alpha_N}}$. Let $W_0^k(\Omega)$ be the closure of $C_0^\infty(\Omega)$ and $W^k(\Omega)$ be the closure of $C^\infty(\Omega)$ in the norm $\|\cdot\|$. Then $W_0^k(\Omega)$ is not dense in $W^k(\Omega)$. Proof to be given.

7.1.6. *The basic unweighted identity.* The key fact underlying the entire L^2 theory for $\bar\partial$ is the next result. It shows how the Dirichlet form (or energy form) for the $\bar\partial$-complex can be re-written as a sum of simpler expressions.

The result will be stated for $(0,1)$-forms. There are no new ideas needed to extend the formula to general (p,q)-forms. At least as long as we remain on domains in $\mathbb{C}^n$.

Proposition 7.13 (Basic estimate I)**.** *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$, with normalized defining function r.*

If $u\in\mathcal{D}^{0,1}$, then

$$\left\|\bar\partial u\right\|^2 + \left\|\bar\partial^* u\right\|^2 = \int_\Omega \sum_{k,j=1}^n \left|\frac{\partial u_j}{\partial \bar z_k}\right|^2 dV + \int_{b\Omega} i\partial\bar\partial r[u,\bar u]\,dS. \tag{7.14}$$

Proof. To be given in detail. □

Some comments about (7.14):

- Only "half" the first derivatives of u – the barred derivatives in the first term on the RHS of (7.14) – are controlled by the LHS of (7.14). This basic *non-ellipticity* distinguishes the $\bar\partial$-complex from, e.g., the d-complex.
- The assumption $r\in C^2\left(\overline{\Omega}\right)$ is needed for IBP. But limiting arguments show inequalities related to (7.14) hold under weaker hypotheses. Done in Lecture 8.
- In order to use duality theorem to solve $\bar\partial$, need RHS of (7.14) to be positive. Notice the natural appearance of Ω being pseudconvex.

On the last comment, note the connection between $D\left(\bar\partial^*\right)$ and the geometry of $b\Omega$:

$$v\in\mathcal{D}_p^{0,1} \;\Leftrightarrow\; (v_1,\dots,v_n)\in T_p^{\mathbb{C}}(b\Omega). \quad \textit{ExerciseI.}$$

A noteworthy consequence of Proposition 7.13 is triviality of *harmonic forms* on strongly pseudoconvex domains. The space of harmonic $(0,1)$-forms on Ω is defined

$$\mathcal{H}^{0,1}(\Omega) = \left\{u\in\mathcal{D}^{0,1} : \bar\partial u = 0 = \bar\partial^* u\right\}.$$

Corollary 7.15. *If $\Omega\subset\mathbb{C}^n$ is strongly pseudoconvex, then $\mathcal{H}^{0,1}(\Omega)=\{0\}$.*

Proof. Let $u=\sum u_j\,d\bar z_j\in\mathcal{H}^{0,1}(\Omega)$. From (7.14)

$$\int_\Omega \sum_{k,j=1}^n \left|\frac{\partial u_j}{\partial \bar z_k}\right|^2 dV + \int_{b\Omega}|u|^2\,dS \le \frac{1}{c}\left(\left\|\bar\partial u\right\|^2 + \left\|\bar\partial^* u\right\|^2\right) = 0,$$

where $i\partial\bar\partial r(u,\bar u) \geq c|u|^2$. Thus each component of u is holomorphic on Ω, and vanishes on $b\Omega$. Consequently $u \equiv 0$. $\square$

7.2. **Hörmander's weighted L^2 estimates.** Let $\lambda \in C^2(\Omega)$. Introducing an L^2 structure involving λ yields flexible estimates on $\bar\partial$ with far-ranging applications.

7.2.1. *Weighted norms.* We continue to use the pointwise inner product (7.1) on forms. For global inner product(s), define

$$(\beta,\gamma)_\lambda = \int_\Omega \langle \beta,\gamma\rangle\, e^{-\lambda}\, dV. \tag{7.16}$$

Denote the associated norm $\|\beta\|_\lambda$. When $\lambda \equiv 0$, continue to denote the inner product and norm without subscript., i.e. $(\cdot,\cdot) = (\cdot,\cdot)_0$. The completion of $\Lambda^{p,q}(\Omega)$ with respect to $\|\cdot\|_\lambda$ is denoted

$$L^2_{p,q}\left(\Omega, e^{-\lambda}\right) = \{(p,q) \text{ forms } \eta : \|\eta\|_\lambda < \infty\}.$$

At this point it is puzzling why the weight is introduced in the form $e^{-\lambda}$. One reason is this allows λ to take both positive and negative values while still having a non-negative measure in the integrals. A more substantial reason is that the basic identity below is simplest with this form of the weight. The *curvature* of the metric $e^{-\lambda}$ (on the trivial line bundle $\mathbb{C}\times\Omega$) arises when we IBP. Possible discussion.

7.2.2. *The weighted adjoint $\bar\partial^*_\lambda$.* The maximal extension of $\bar\partial$ is taken as before; the set $D(\bar\partial)$ is unchanged by the weighted L^2 structure, as long as $\lambda \in L^\infty(\Omega)$. However the adjoint operator changes significantly. Parallel to (7.4), the domain of $\bar\partial^*_\lambda$ is

$$D\left(\bar\partial^*_\lambda\right) = \left\{v \in L^2_{p,q}(\Omega) : \exists C > 0 \text{ such that } \left|\left(\bar\partial u, v\right)_\lambda\right| \leq C\|u\|_\lambda \text{ for all } u \in D\left(\bar\partial\right)\right\} \tag{7.17}$$

To do computations, a version of Proposition 7.11 is needed. Note a restriction on the weight is stipulated in the following result

Proposition 7.18. *Suppose that $\lambda \in C^1\left(\overline{\Omega}\right)$. Then $\mathcal{D}^{p,q}$ is dense in $D\left(\bar\partial^*_\lambda\right)$ in the norm*

$$\|u\|_{\mathcal{G},\lambda} =: \|u\|_\lambda + \left\|\bar\partial u\right\|_\lambda + \left\|\bar\partial^*_\lambda u\right\|_\lambda.$$

Remark 7.19. Deeper waters if $\lambda \notin C^1\left(\overline{\Omega}\right)$. Possible discussion.

What about computing $\bar\partial^*_\lambda$? Again specialize to $(0,1)$-forms: let $v = \sum_{k=1}^n v_k\, d\bar z_k \in \mathcal{D}^{0,1}$. The elementary $(a,b)_w = (a, e^{-w}\, b)$ and the computation that gave (7.10) yields

$$\begin{aligned}\bar\partial^*_\lambda v &= -e^\lambda \sum_{k=1}^n \frac{\partial}{\partial z_k}\left(e^{-\lambda} v_k\right)\\ &= \bar\partial^* v + \sum_{k=1}^n \frac{\partial\lambda}{\partial z_k}\, v_k.\end{aligned} \tag{7.20}$$

7.2.3. *The basic weighted identity.*

Proposition 7.21 (Basic identity II)**.** *Let Ω be a smoothly bounded domain in $\mathbb{C}^n$, with normalized defining function r. Suppose that $\lambda \in C^2(\Omega) \cap C^1\left(\overline{\Omega}\right)$.*

If $u \in \mathcal{D}^{0,1}$, then

$$\begin{aligned}\left\|\bar\partial u\right\|^2_\lambda + \left\|\bar\partial^*_\lambda u\right\|^2_\lambda &= \int_\Omega \sum_{k,j=1}^n \left|\frac{\partial u_j}{\partial \bar z_k}\right|^2 e^{-\lambda}\, dV\\ &\quad + \int_\Omega i\partial\bar\partial\lambda[u,\bar u] e^{-\lambda}\, dV + \int_{b\Omega} i\partial\bar\partial r[u,\bar u] e^{-\lambda}\, dS.\end{aligned} \tag{7.22}$$

Proof. Discussed now. Details when twisted identity derived. □

Need to extract a **positive** lower bound on $\left\|\bar{\partial} u\right\|_{\lambda}^{2}+\left\|\bar{\partial}_{\lambda}^{*} u\right\|_{\lambda}^{2}$ if aim to use Basic Duality theorem to solve $\bar{\partial}$. The shape of (7.22) directly leads to two basic notions in complex analysis already discussed: pseudoconvexity of $b\Omega$ and plurisubharmonicity of λ ensure the last 2 terms in (7.22) are non-negative.

This leads to our initial $\bar{\partial}$ theorem:

Theorem 7.23 (Basic Hörmander)**.** *Let Ω be a smoothly bounded pseudoconvex domain and λ be strongly plurisubharmonic on $\overline{\Omega}$.*

Then, for any $\alpha \in L_{0,1}^{2}\left(\Omega, e^{-\lambda}\right)$ with $\bar{\partial}\alpha = 0$, $\exists\, u$ such that $\bar{\partial} u = \alpha$ and

$$\int_{\Omega}|u|^{2} e^{-\lambda} \leq \frac{1}{C} \int_{\Omega}|\alpha|^{2} e^{-\lambda}, \tag{7.24}$$

where $C > 0$ is a lower bound on $i\partial\bar{\partial}\lambda$ on $\overline{\Omega}$.

Proof. ExerciseII, using Basic Duality lemma. □