7. \( L^2 \) estimates on \( \bar{\partial} \): Part I

Throughout \( \Omega \subset \mathbb{C}^n \) is a smoothly bounded domain (usually \( b\Omega \in C^2 \) suffices). The goal of this lecture and the next is to unpack the basic \( L^2 \) estimates that hold on \( \bar{\partial} \). This is done in 3 passes: (i) for the usual \( L^2 \) structure on \( \mathbb{C}^n \), (ii) for weighted \( L^2 \) norms – i.e., the celebrated Hörmander estimates, and (iii) for weighted \( L^2 \) norms on a twisted version of the \( \bar{\partial} \)-complex. The twisted-weighted estimates give results not accessible by Hörmander estimates – this will be clearer in applications discussed afterwards.

7.1. Estimates in ordinary \( L^2 \). The \( \bar{\partial} \) operator in Section 3.1 is not \textit{a priori} a Hilbert space operator. However it can be extended to such an operator, given its initial definition on \( \Lambda^{p,q}(\Omega) \).

7.1.1. Norms of forms. First need pointwise inner products and norms on forms. If \( \beta, \gamma \in \Lambda^{p,q}(\Omega) \) is expressed in the form (3.1), i.e.,

\[
\beta = \sum'_{|I|=p, |J|=q} \beta_{IJ} dz^I \wedge d\bar{z}^J \quad \text{and} \quad \gamma = \sum'_{|K|=p, |L|=q} \gamma_{KL} dz^K \wedge d\bar{z}^L,
\]

define

\[
\langle \beta, \gamma \rangle = \sum_{|I|=p, |J|=q} \beta_{IJ} \bar{\gamma}_{IJ}.
\]

(7.1)

Forms of different bi-degrees are simply declared to be pointwise perpendicular, i.e. if \( \mu \in \Lambda^{p,q}(\Omega), \nu \in \Lambda^{r,s}(\Omega) \) and \( (p,q) \neq (r,s) \), then \( \langle \mu, \nu \rangle = 0 \). More generally, notice that \( \langle \beta, \gamma \rangle \) in (7.1) only involves matching indices of \( \beta \) and \( \gamma \).

A global inner product is defined by integration:

\[
(\beta, \gamma) = \int_{\Omega} \langle \beta, \gamma \rangle \ dV,
\]

(7.2)

where \( dV \) is the euclidean volume form. The associated \( L^2 \)-norm is denoted \( \| \beta \| = (\beta, \beta)^{\frac{1}{2}} \).

\( \Lambda^{p,q}(\Omega) \) is not complete in \( \| \cdot \| \). Let \( L^2_{p,q}(\Omega) \) denote the completion of \( \Lambda^{p,q}(\Omega) \) with respect to \( \| \cdot \| \). A standard measure theory argument gives

\[
L^2_{p,q}(\Omega) = \{ (p,q) \text{ forms } \eta \text{ with measurable components : } \| \eta \| < \infty \}.
\]

7.1.2. Extending \( \bar{\partial} \). To turn \( \bar{\partial} \) into an \( L^2 \) operator, define

\[
D(\bar{\partial}) = \{ u \in L^2_{p,q}(\Omega) : \exists \{ u_j \} \in \Lambda_{p,q}(\Omega) \text{ such that } \lim u_j = u \text{ and } \{ \bar{\partial} u_j \} \text{ is Cauchy in } L^2_{p,q+1}(\Omega) \}.
\]

(7.3)

\( D(\bar{\partial}) \) is the domain of \( \bar{\partial} \). This set of forms can also be viewed as

\[
D(\bar{\partial}) = \{ u \in L^2_{p,q}(\Omega) : \bar{\partial} u \in L^2_{p,q+1}(\Omega) \}.
\]
where \( \bar{\partial} \) acts on \( u \) as a distributional derivative. Exercise II.

For \( u \in D(\bar{\partial}) \), set

\[
\bar{\partial} u = \lim \bar{\partial} u_j.
\]

We point out an abuse of notation here. \( \bar{\partial} u_j \) appearing on the RHS is well-defined since \( u_j \) is smooth, but the equation defines the meaning of \( \bar{\partial} u \) for \( u \in D(\bar{\partial}) \). The symbol \( \bar{\partial} \), which denoted a pointwise-defined operator is used here differently. Notational abuse like this is widespread; it is also confusing (at first)! In any case, verify the definition given of \( \bar{\partial} u \) is independent of \( \{ u_j \} \) Exercise I. Thus the equation defines an operator \( \bar{\partial} : D(\bar{\partial}) \rightarrow L^2(\cdot,\cdot)(\Omega) \), which coincides with the earlier definition of \( \bar{\partial} v \) when \( v \in \Lambda^{p,q}(\Omega) \).

The pair \( (\bar{\partial}, D(\bar{\partial})) \) is the maximal extension of \( \bar{\partial} \), the operator \( \bar{\partial} \) is densely defined on the \( L^2 \) spaces, and \( \bar{\partial} \) is a closed operator. But \( \bar{\partial} \) is not a bounded operator on \( L^2 \). Give example, here or in Lecture 6. Lecture 6 discusses all this in more detail.

From now on, \( \bar{\partial} \) denotes the extended operator above.

7.1.3. \( \bar{\partial}^* \) and its domain. Since \( \bar{\partial} \) is closed and densely defined, it has a well-defined adjoint operator, \( \bar{\partial}^* \). The adjoint operator is also unbounded, but is a closed operator on the domain given by

\[
D(\bar{\partial}^*) = \{ v \in L^2_{p,q}(\Omega) : \exists C > 0 \text{ such that } |(\bar{\partial} u, v)| \leq C\|u\| \text{ for all } u \in D(\bar{\partial}) \} \tag{7.4}
\]

To define \( \bar{\partial}^* \) on \( D(\bar{\partial}^*) \), note \( [i,4] \) says \( u \rightarrow (\bar{\partial} u, v) \) is a bounded linear functional on \( D(\bar{\partial}) \). This functional extends to entire \( L^2 \) space since \( D(\bar{\partial}) \) is dense. The Riesz representative of this functional is, by definition, \( \bar{\partial}^* v \).

Thus the usual adjoint relationship holds

\[
(u, \bar{\partial}^* v) = (\bar{\partial} u, v) \quad \forall u \in D(\bar{\partial}) , v \in D(\bar{\partial}^*) . \tag{7.5}
\]

To do serious computations, two elements are needed: (i) an expression of \( \bar{\partial}^* \) as a differential operator, and (ii) a (dense) subspace of \( D(\bar{\partial}^*) \) where calculus can be applied.

7.1.4. On integration by parts. To belong to \( D(\bar{\partial}^*) \), a form must satisfy conditions “on” \( b\Omega \): terms arising during integration by parts must vanish in a suitable sense. For a general form, these conditions are abstract and difficult to characterize. But for forms that are smooth up to \( b\Omega \), the boundary conditions become concrete and relatively simple.

Let \( r \) be a defining function for \( \Omega \). Recall the divergence theorem: if \( \Omega \subset \mathbb{R}^N \) is a smoothly bounded domain, \( F = (F_1,\ldots,F_N) \) is a smooth vector field, and \( \nu \) is the outward unit normal field to \( b\Omega \), then

\[
\int_\Omega \nabla \cdot F \, dV = \int_{b\Omega} F \cdot \nu \, dS ,
\]

where \( dS \) is surface measure on \( b\Omega \). Let \((x_1, \ldots, x_{2n})\) be the underlying real coordinates on \( \mathbb{C}^n \) in any fixed order. If \( X = \sum c_i \frac{\partial}{\partial x_i} \) is a first order differential operator with coefficients
\( c_i \in C^\infty(\Omega) \) and \( a, b \in C^\infty(\Omega) \), it follows
\[
(Xa, b) = \int_\Omega \sum_{i=1}^{2n} c_i \frac{\partial a}{\partial x_i} \bar{b} \, dV
\]
\[
= \int_\Omega \sum_{i=1}^{2n} \frac{\partial}{\partial x_i} (c_i \bar{a} b) \, dV - \int_\Omega a \frac{\partial}{\partial x_i} (c_i \bar{b}) \, dV
\]
\[
= \int_\Omega \nabla \cdot (c_1 \bar{a} b, \ldots, c_{2n} \bar{a} b) \, dV + (a, X'b)
\]
\[
= (a, X'b) + \int_{\partial \Omega} \sum_{i=1}^{2n} c_i \frac{\partial r}{\partial x_i} \bar{a} \bar{b} \, dS |\nabla r|.
\] (7.6)

The last equality defines the formal adjoint \( X' \) of \( X \).

The same considerations apply to forms, and to form-valued operators on forms (like \( \bar{\partial} \)). This is true because we have extended the \( L^2 \) inner product on functions “by linearity” to our spaces of forms.

A specialized defining functions simplifies (7.6), which is occasionally useful. Say a defining function \( \rho \) for a smoothly bounded domain \( \Omega \) is normalized if \( \|\partial \rho\| = 1 \) on \( \partial \Omega \). Can always find a normalized defining function Exercise0.. For the Cauchy-Riemann operators IBP can be written: if \( \rho \) is a normalized defining function for \( \Omega \)
\[
(\frac{\partial}{\partial \bar{z}_k}, f, g) = - (f, \frac{\partial g}{\partial \bar{z}_k}) + \int_{\partial \Omega} f \bar{g} \frac{\partial \rho}{\partial \bar{z}_k} \, dS,
\] (7.7)
for \( f, g \in \Lambda^{0,0}(\Omega) \). Similar expressions hold for forms.

7.1.5. The space \( D^{p,q} \). Now consider smooth up-to-the-boundary forms in \( D (\bar{\partial}^*) \). Let
\[
D^{p,q} = D (\bar{\partial}^*) \cap \Lambda^{p,q}(\Omega).
\]
It is easy to show that

**Proposition 7.8.** A form \( u = \sum u_{I,M}dz^I \wedge d\bar{z}^M \in D^{p,q} \) if and only if
\[
\sum_{k=1}^{n} u_{I,k,J}(z) \frac{\partial r}{\partial \bar{z}_k} (z) = 0 \quad \text{for all } |I| = p, |J| = q - 1, \text{ and } z \in \partial \Omega.
\] (7.9)

**Proof.** Focus on the boundary integral(s) arising from (7.7). ExerciseII. \( \square \)

Specialize to \((0,1)\)-forms. Then
\[
D^{0,1} = \left\{ v \in \Lambda^{0,1}(\Omega) : \sum_{k=1}^{n} \frac{\partial r}{\partial \bar{z}_k} v_k = 0 \text{ on } \partial \Omega \right\}.
\]

Moreover, if \( v = \sum_{k=1}^{n} v_k \, d\bar{z}_k \in D^{0,1} \), (7.7) and (7.5) imply
\[
\bar{\partial}^* v = - \sum_{k=1}^{n} \frac{\partial v_k}{\partial \bar{z}_k}.
\] (7.10)

**Exercisel.** The operator on the RHS of (7.10) is the formal adjoint of \( \bar{\partial} \) and denoted \( \vartheta v \). \( \vartheta v \) is defined (as a differential operator) for any smooth \( v \); the computation just given shows \( \bar{\partial}^* v = \vartheta v \) when \( v \in D^{0,1} \).

This is useful because
Proposition 7.11. \( \mathcal{D}^{p,q} \) is dense in \( D(\bar{\partial}^*) \cap D(\bar{\partial}) \) in the norm
\[
\|u\|_\mathcal{G} := \|u\| + \|\bar{\partial}u\| + \|\bar{\partial}^* u\|.
\]

Proof. The proof is a bit intricate. It hinges on the basic Friedrich mollifier technique, but the tangential and “normal” components of a form \( u \) must be handled separately. See [?] pages 94–98 or [?] pages 70–74 of a detailed proof.

\( \square \)

Remark 7.12. \( \Lambda_0^{p,q}(\Omega) \) is not dense in \( D(\bar{\partial}^*) \cap D(\bar{\partial}) \) in the graph norm \( \| \cdot \|_\mathcal{G} \).

The main issue can be seen with ordinary Sobolev spaces. For \( \Omega \subset \mathbb{R}^N, \ k \in \mathbb{Z}^+ \), and \( f \in C^k(\Omega) \), define the norm
\[
\|f\|_\mathcal{G}^2 = \sum_{\|\alpha\| \leq k} \|D^\alpha f\|^2,
\]
where \( \| \cdot \| \) is the \( L^2 \)-norm and \( D^\alpha = \frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_N^{\alpha_N}} \). Let \( W_0^k(\Omega) \) be the closure of \( C^\infty_0(\Omega) \) and \( W^k(\Omega) \) be the closure of \( C^\infty(\Omega) \) in the norm \( \| \cdot \| \). Then \( W_0^k(\Omega) \) is not dense in \( W^k(\Omega) \).

Proof to be given.

7.1.6. The basic unweighted identity. The key fact underlying the entire \( L^2 \) theory for \( \bar{\partial} \) is the next result. It shows how the Dirichlet form (or energy form) for the \( \bar{\partial} \)-complex can be re-written as a sum of simpler expressions.

The result will be stated for \( (0,1) \)-forms. There are no new ideas needed to extend the formula to general \( (p,q) \)-forms. At least as long as we remain on domains in \( \mathbb{C}^n \).

Proposition 7.13 (Basic estimate I). Let \( \Omega \) be a smoothly bounded domain in \( \mathbb{C}^n \), with normalized defining function \( r \).

If \( u \in \mathcal{D}^{0,1} \), then
\[
\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 = \int_\Omega \sum_{k,j=1}^n \left| \frac{\partial u_j}{\partial \bar{z}_k} \right|^2 dV + \int_{\partial \Omega} i\bar{\partial}r[u, \bar{u}] dS. \tag{7.14}
\]

Proof. To be given in detail.

\( \square \)

Some comments about (7.14):

- Only “half” the first derivatives of \( u \) – the barred derivatives in the first term on the RHS of (7.14) – are controlled by the LHS of (7.14). This basic non-ellipticity distinguishes the \( \bar{\partial} \)-complex from, e.g., the \( d \)-complex.
- The assumption \( r \in C^2(\bar{\Omega}) \) is needed for IBP. But limiting arguments show inequalities related to (7.14) hold under weaker hypotheses. Done in Lecture 8.
- In order to use duality theorem to solve \( \bar{\partial} \), need RHS of (7.14) to be positive. Notice the natural appearance of \( \Omega \) being pseudoconvex.

On the last comment, note the connection between \( D(\bar{\partial}^*) \) and the geometry of \( b\Omega \):

\( v \in \mathcal{D}_p^{0,1} \iff (v_1, \ldots, v_n) \in T^C_p(\bar{b}\Omega). \) \quad \text{Exercise I.}

A noteworthy consequence of Proposition 7.13 is triviality of harmonic forms on strongly pseudoconvex domains. The space of harmonic \( (0,1) \)-forms on \( \Omega \) is defined
\[
\mathcal{H}^{0,1}(\Omega) = \{ u \in \mathcal{D}^{0,1} : \bar{\partial} u = 0 = \bar{\partial}^* u \}.
\]

Corollary 7.15. If \( \Omega \subset \mathbb{C}^n \) is strongly pseudoconvex, then \( \mathcal{H}^{0,1}(\Omega) = \{ 0 \} \).

Proof. Let \( u = \sum u_j \, dz_j \in \mathcal{H}^{0,1}(\Omega) \). From (7.14)
\[
\int_\Omega \sum_{k,j=1}^n \left| \frac{\partial u_j}{\partial \bar{z}_k} \right|^2 dV + \int_{\partial \Omega} |u|^2 dS \leq \frac{1}{c} \left( \|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2 \right) = 0,
\]
where \( i \partial \bar{\partial} r(u, \bar{u}) \geq c|u|^2 \). Thus each component of \( u \) is holomorphic on \( \Omega \), and vanishes on \( \partial \Omega \). Consequently \( u \equiv 0 \).

7.2. **Hörmander’s weighted \( L^2 \) estimates.** Let \( \lambda \in C^2(\Omega) \). Introducing an \( L^2 \) structure involving \( \lambda \) yields flexible estimates on \( \partial \) with far-ranging applications.

7.2.1. **Weighted norms.** We continue to use the pointwise inner product \([7.1]\) on forms. For global inner product(s), define

\[
(\beta, \gamma)_\lambda = \int_\Omega (\beta, \gamma) e^{-\lambda} dV.
\]

(7.16)

Denote the associated norm \( \| \beta \|_\lambda \). When \( \lambda \equiv 0 \), continue to denote the inner product and norm without subscript, i.e. \( (\cdot, \cdot) = (\cdot, \cdot)_0 \). The completion of \( \Lambda^{p, \beta}(\Omega) \) with respect to \( \| \cdot \|_\lambda \) is denoted

\[
L^2_{p, q}(\Omega, e^{-\lambda}) = \{ (p, q) \text{ forms } \eta : \| \eta \|_\lambda < \infty \}.
\]

At this point it is puzzling why the weight is introduced in the form \( e^{-\lambda} \). One reason is this allows \( \lambda \) to take both positive and negative values while still having a non-negative measure in the integrals. A more substantial reason is that the basic identity below is simplest with this form of the weight. The *curvature* of the metric \( e^{-\lambda} \) (on the trivial line bundle \( \mathbb{C} \times \Omega \)) arises when we IBP. Possible discussion.

7.2.2. **The weighted adjoint \( \bar{\partial}_\lambda^* \).** The maximal extension of \( \bar{\partial} \) is taken as before; the set \( D(\bar{\partial}) \) is unchanged by the weighted \( L^2 \) structure, as long as \( \lambda \in L^\infty(\Omega) \). However the adjoint operator changes significantly. Parallel to \([7.4]\), the domain of \( \bar{\partial}_\lambda^* \) is

\[
D(\bar{\partial}_\lambda^*) = \{ v \in L^2_{p, q}(\Omega) : \exists C > 0 \text{ such that } |(\bar{\partial} u, v)_\lambda| \leq C \| u \|_\lambda \text{ for all } u \in D(\bar{\partial}) \}. \tag{7.17}
\]

To do computations, a version of Proposition \([7.11]\) is needed. Note a restriction on the weight is stipulated in the following result

**Proposition 7.18.** Suppose that \( \lambda \in C^1(\overline{\Omega}) \). Then \( \mathcal{D}^{p, q} \) is dense in \( D(\bar{\partial}_\lambda^*) \) in the norm

\[
\| u \|_{g, \lambda} := \| u \|_\lambda + \| \bar{\partial} u \|_\lambda + \| \bar{\partial}_\lambda^* u \|_\lambda.
\]

**Remark 7.19.** Deeper waters if \( \lambda \notin C^1(\overline{\Omega}) \). Possible discussion.

What about computing \( \bar{\partial}_\lambda^* \)? Again specialize to \((0, 1)\)-forms: let \( v = \sum_{k=1}^n v_k d\bar{z}_k \in \mathcal{D}^{0, 1} \). The elementary \( (a, b)_w = (a, e^{-w} b) \) and the computation that gave \([7.10]\) yields

\[
\bar{\partial}_\lambda^* v = -e^\lambda \sum_{k=1}^n \frac{\partial}{\partial z_k} \left( e^{-\lambda} v_k \right)
= \bar{\partial} v + \sum_{k=1}^n \frac{\partial \lambda}{\partial z_k} v_k. \tag{7.20}
\]

7.2.3. **The basic weighted identity.**

**Proposition 7.21** (Basic identity II). Let \( \Omega \) be a smoothly bounded domain in \( \mathbb{C}^n \), with normalized defining function \( r \). Suppose that \( \lambda \in C^2(\Omega) \cap C^1(\overline{\Omega}) \).

If \( u \in \mathcal{D}^{0, 1} \), then

\[
\| \bar{\partial} u \|^2_\lambda + \| \bar{\partial}_\lambda^* u \|^2_\lambda = \int_\Omega \sum_{k,j=1}^n \left| \frac{\partial u_j}{\partial \bar{z}_k} \right|^2 e^{-\lambda} dV
+ \int_\Omega i\partial \bar{\partial} \lambda [u, \bar{u}] e^{-\lambda} dV + \int_{\partial \Omega} i\partial \bar{\partial} r [u, \bar{u}] e^{-\lambda} dS. \tag{7.22}
\]
Proof. Discussed now. Details when twisted identity derived. □

Need to extract a positive lower bound on \( \| \bar{\partial} u \|_\lambda^2 + \| \bar{\partial}_\lambda^* u \|_\lambda^2 \) if aim to use Basic Duality theorem to solve \( \bar{\partial} \). The shape of (7.22) directly leads to two basic notions in complex analysis already discussed: pseudoconvexity of \( b\Omega \) and plurisubharmonicity of \( \lambda \) ensure the last 2 terms in (7.22) are non-negative.

This leads to our initial \( \bar{\partial} \) theorem:

**Theorem 7.23** (Basic Hörmander). Let \( \Omega \) be a smoothly bounded pseudoconvex domain and \( \lambda \) be strongly plurisubharmonic on \( \Omega \).

Then, for any \( \alpha \in L^2_{0,1}(\Omega, e^{-\lambda}) \) with \( \bar{\partial} \alpha = 0 \), \( \exists u \) such that \( \bar{\partial} u = \alpha \) and

\[
\int_{\Omega} |u|^2 e^{-\lambda} \leq \frac{1}{C} \int_{\Omega} |\alpha|^2 e^{-\lambda},
\]

(7.24)

where \( C > 0 \) is a lower bound on \( i\partial \bar{\partial} \lambda \) on \( \overline{\Omega} \).

**Proof.** ExerciseII, using Basic Duality lemma. □