MSRI 2018: THE $\overline{\partial}$-PROBLEM IN THE TWENTY-FIRST CENTURY

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Homework-3

(1) *Worm domain.* Let $\beta > \pi/2$ be a real number. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function with the following properties:
   (i) η is a non-negative, even and convex function.
   (ii) $\eta^{-1}(0) = [-\beta + \pi/2, \beta - \pi/2] := I_\beta$.
   (iii) there exists an $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$.
   (iv) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

   Let Ω_β be the domain

$$\Omega_\beta = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 + e^{i \log |z_2|^2} \right|^2 < 1 - \eta(\log |z_2|^2) \right\}.$$

   For each $\beta > \pi/2$, Ω_β is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$.
   (a) To show that Ω_β is smooth, it suffices to show that $\nabla \rho(z) \neq 0$ at each $z \in b\Omega_\beta$, where

$$\rho(z_1, z_2) = \left| z_1 + e^{i \log |z_2|^2} \right|^2 - 1 + \eta(\log |z_2|^2)$$

   is the defining function for Ω_β. Show that if $(\partial \rho / \partial z_1)(z) = 0$ for some $z \in b\Omega_\beta$ then $(\partial \rho / \partial z_2)(z) \neq 0$.
   (b) Show that locally, Ω_β can be defined by

$$|z_1|^2 e^{\arg z_2^2} + 2\mathrm{Re}(z_1 e^{-i \log z_2^2}) + \eta(\log |z_2|^2) e^{\arg z_2^2} < 0. \tag{3.1}$$

   (c) Show that the first term in (3.1) is a plurisubharmonic function. The second term is pluriharmonic as it is the real part of a holomorphic function. Show that the third term is plurisubharmonic by computing its Levi form.
   (d) When $\beta \geq 3\pi/2$, the worm domain Ω_β does not have a basis of pseudoconvex neighbourhoods. Note that $\overline{\Omega}_\beta$ contains the set

$$K = \left\{ (0, z_2) \in \mathbb{C}^2 : -\pi \leq \log |z_2|^2 \leq \pi \right\} \bigcup \left\{ (z_1, z_2) \in \mathbb{C}^2 : \log |z_2|^2 = \pi \text{ or } -\pi, |z_1 - 1| < 1 \right\},$$

   when $\beta \geq 3\pi/2$. Show that any holomorphic function in a neighbourhood of K extends holomorphically to the set

$$\widehat{K} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : -\pi \leq \log |z_2|^2 \leq \pi, |z_1 - 1| < 1 \right\}.$$

(2) *Non-solvable $\bar{\partial}$ equation.* If $\alpha \in \Lambda^{0,1}(\mathbb{D}^2)$ is a $\overline{\partial}$-closed form then the equation

$$\overline{\partial} u = \alpha$$

   is solvable. This will be proved in the upcoming lectures. Feel free to use this fact.
   Let

$$U_1 = \left\{ z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < \frac{1}{2} \right\}, \quad U_2 = \left\{ z \in \mathbb{C}^2 : \frac{1}{2} < |z_1| < 1, |z_2| < 1 \right\}$$

   and set $\Omega = U_1 \cup U_2$.

(a) Let $a \in \mathbb{C}$ be such that $\frac{1}{2} < |a| < 1$, and let $\phi \in \mathcal{C}_0^\infty(\mathbb{C})$ be a cutoff such that $\phi \equiv 1$ in a neighborhood of a, $\phi \equiv 0$ in a neighborhood of $\{|z| \leq \frac{1}{2}\}$ and ϕ is supported in the unit disc $\{|z| < 1\}$. Show that

$$\eta = \begin{cases} 0 & \text{on } U_1 \\ \bar{\partial}\left(\frac{1-\phi(z_2)}{z_1(z_2-a)}\right) & \text{on } U_2 \end{cases}$$

is a well-defined $(0,1)$-form on Ω, and $\bar{\partial}\eta = 0$. We will show that there is no $u \in \mathcal{C}^\infty(\Omega)$ such that $\bar{\partial}u = \eta$.

(b) Show that a if a solution u to $\bar{\partial}u = \eta$ exists. Then there are holomorphic functions $h_1 \in \mathcal{O}(U_1)$ and $h_2 \in \mathcal{O}(U_2)$ such that

$$\frac{1}{z_1(z_2-a)} = h_1 - h_2 \tag{3.2}$$

Hint: If there exist h_1 and h_2 as above, then

$$u|_{U_1} = h_1 \tag{3.3}$$

and

$$u|_{U_2} = \frac{1-\phi(z_2)}{z_1(z_2-a)} + h_2. \tag{3.4}$$

Since $\phi(z_2) = 0$ for $(z_1, z_2) \in U_1$, it follows that on $U_1 \cap U_2$, we have (3.2).

(c) Conversely, suppose that there exist holomorphic $h_j \in \mathcal{O}(U_j)$ such that (3.2) holds. Show that there exist $u \in C^\infty(\Omega)$ such that $\bar{\partial}u = \eta$.
Hint: If we define u using (3.3) and (3.4), the function u is well-defined and satisfies the equation $\bar{\partial}u = \eta$.

(d) Show that there is no $h_j \in \mathcal{O}(U_j)$, $j = 1, 2$ such that (3.2) holds. It follows that the equation $\bar{\partial}u = \eta$ has no solution.
Hint: For a contradiction, assume that there are $h_j \in \mathcal{O}(U_j)$ such that (3.2) holds. Since h_1 is holomorphic on the bidisc $\{|z_1| < 1\} \times \{|z_2| < \frac{1}{2}\}$, it has a Taylor expansion of the form $h_1(z_1, z_2) = \sum_{m,n=0}^\infty a_{mn} z_1^m z_2^n$. And h_2 is holomorphic in the product $\{\frac{1}{2} < |z_1| < 1\} \times \{|z_2| < 1\}$ of an annulus and a disc, so it has a Laurent expansion of the form $h_2(z_1, z_2) = \sum_{m=-\infty}^\infty \sum_{n=0}^\infty b_{mn} z_1^m z_2^n$. The Laurent expansion of $\frac{1}{z_1(z_2-a)}$ on the product $U_1 \cap U_2 = \{\frac{1}{2} < |z_1| < 1\} \times \{|z_2| < 1\}$ is $\sum_{\nu=0}^\infty a^{-(\nu+1)} z_1^{-1} z_2^\nu$. Therefore from (3.2) and the uniqueness of Laurent expansions we see that we must have on $U_1 \cap U_2$: $h_1 = 0$, and $h_2(z_1, z_2) = -\frac{1}{z_1(z_2-a)}$. But then h_2 does not extend to U_2 holomorphically because of the pole along $z_2 = a$. By the uniqueness of analytic continuation, there can be no holomorphic $h_j \in \mathcal{O}(U_j)$ satisfying (3.2).

(3) *Canonical representation of domains in $\mathbb{C}^n$:* Let $\Omega \in \mathbb{C}^n$ and $\partial\Omega$ is C^2.
(a) Near any point $z^0 \in \partial\Omega$, we can introduce holomorphic coordinates $(z_1, \ldots, z_n)$ (the change of variables are given by holomorphic functions) centered at z^0 so that

$$\Omega = \{\mathrm{Im}(z_n) > \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + E(z)\} \tag{3.5}$$

Here the λ_j are real numbers, $z_j = x_j + iy_j$, and $E(z) = x_n l(z') + Dx_n^2 + o(|z|^2)$, as $z \to 0$. Also $l(z')$ is a linear function of $x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1}$ and D is a real number.
(b)With an additional change of variables, the above representation can be simplified to state

$$y_n = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + o(|z'|^2), \qquad \text{for } z' \to 0$$