(1) **Worm domain.** Let $\beta > \pi/2$ be a real number. Let $\eta : \mathbb{R} \to \mathbb{R}$ be a fixed smooth function with the following properties:

(i) $\eta$ is a non-negative, even and convex function.

(ii) $\eta^{-1}(0) = [-\beta + \pi/2, \beta - \pi/2] := I_\beta$.

(iii) there exists an $a > 0$ such that $\eta(x) > 1$ if $x < -a$ or $x > a$.

(iv) $\eta'(x) \neq 0$ if $\eta(x) = 1$.

Let $\Omega_\beta$ be the domain

$$
\Omega_\beta = \left\{ (z_1, z_2) \in \mathbb{C}^2 : \left| z_1 + e^{i \log |z_2|^2} \right|^2 < 1 - \eta(\log |z_2|^2) \right\}.
$$

For each $\beta > \pi/2$, $\Omega_\beta$ is a smooth bounded pseudoconvex domain in $\mathbb{C}^2$.

(a) To show that $\Omega_\beta$ is smooth, it suffices to show that $\nabla \rho(z) \neq 0$ at each $z \in b\Omega_\beta$, where $\rho(z_1, z_2) = \left| z_1 + e^{i \log |z_2|^2} \right|^2 - 1 + \eta(\log |z_2|^2)$ is the defining function for $\Omega_\beta$. Show that if $(\partial \rho / \partial z_1)(z) = 0$ for some $z \in b\Omega_\beta$ then $(\partial \rho / \partial z_2)(z) \neq 0$.

(b) Show that locally, $\Omega_\beta$ can be defined by

$$
|z_1|^2 e^{\text{arg} z_2^2} + 2 \text{Re}(z_1 e^{-i \log z_2^2}) + \eta(\log |z_2|^2) e^{\text{arg} z_2^2} < 0.
$$

(3.1)

(c) Show that the first term in (3.1) is a plurisubharmonic function. The second term is pluriharmonic as it is the real part of a holomorphic function. Show that the third term is plurisubharmonic by computing its Levi form.

(d) When $\beta \geq 3\pi/2$, the worm domain $\Omega_\beta$ does not have a basis of pseudoconvex neighbourhoods. Note that $\Omega_\beta$ contains the set

$$
K = \left\{ (0, z_2) \in \mathbb{C}^2 : -\pi \leq \log |z_2|^2 \leq \pi \right\} \cup \left\{ (z_1, z_2) \in \mathbb{C}^2 : \log |z_2|^2 = \pi \text{ or } -\pi, |z_1| = 1, |z_1 - 1| < 1 \right\},
$$

when $\beta \geq 3\pi/2$. Show that any holomorphic function in a neighbourhood of $K$ extends holomorphically to the set

$$
\tilde{K} = \left\{ (z_1, z_2) \in \mathbb{C}^2 : -\pi \leq \log |z_2|^2 \leq \pi, |z_1| = 1, |z_1 - 1| < 1 \right\}.
$$

(2) **Non-solvable $\overline{\partial}$ equation.** If $\alpha \in \Lambda^{0,1}(\mathbb{D}^2)$ is a $\overline{\partial}$-closed form then the equation

$$
\overline{\partial} u = \alpha
$$

is solvable. This will be proved in the upcoming lectures. Feel free to use this fact.

Let

$$
U_1 = \left\{ z \in \mathbb{C}^2 : |z_1| < 1, |z_2| < \frac{1}{2} \right\}, \quad U_2 = \left\{ z \in \mathbb{C}^2 : \frac{1}{2} < |z_1| < 1, |z_2| < 1 \right\}
$$

and set $\Omega = U_1 \cup U_2$. 

(a) Let \( a \in \mathbb{C} \) be such that \( \frac{1}{2} < |a| < 1 \), and let \( \phi \in C^\infty_0(\mathbb{C}) \) be a cutoff such that \( \phi \equiv 1 \) in a neighborhood of \( a \), \( \phi \equiv 0 \) in a neighborhood of \( \{|z| \leq \frac{1}{2}\} \) and \( \phi \) is supported in the unit disc \( \{|z| < 1\} \). Show that

\[
\eta = \begin{cases} 
0 & \text{on } U_1 \\
\frac{1}{\bar{z}_1(z_2-a)} & \text{on } U_2 
\end{cases}
\]

is a well-defined \((0,1)\)-form on \( \Omega \), and \( \partial \eta = 0 \). We will show that there is no \( u \in C^\infty(\Omega) \) such that \( \partial u = \eta \).

(b) Show that if a solution \( u \) to \( \partial u = \eta \) exists. Then there are holomorphic functions \( h_1 \in \mathcal{O}(U_1) \) and \( h_2 \in \mathcal{O}(U_2) \) such that

\[
\frac{1}{z_1(z_2-a)} = h_1 - h_2.
\]  

(3.2)

Hint: If there exist \( h_1 \) and \( h_2 \) as above, then

\[
u \big|_{U_1} = h_1
\]

and

\[
u \big|_{U_2} = \frac{1 - \phi(z_2)}{z_1(z_2-a)} + h_2.
\]  

(3.4)

Since \( \phi(z_2) = 0 \) for \( (z_1,z_2) \in U_1 \), it follows that on \( U_1 \cap U_2 \), we have \( (3.2) \).

(c) Conversely, suppose that there exist holomorphic \( h_j \in \mathcal{O}(U_j) \) such that \( (3.2) \) holds. Show that there exist \( u \in C^\infty(\Omega) \) such that \( \partial u = \eta \).

Hint: If we define \( u \) using \( (3.3) \) and \( (3.4) \), the function \( u \) is well-defined and satisfies the equation \( \partial u = \eta \).

(d) Show that there is no \( h_j \in \mathcal{O}(U_j) \), \( j = 1,2 \) such that \( (3.2) \) holds. It follows that the equation \( \partial u = \eta \) has no solution.

Hint: For a contradiction, assume that there are \( h_j \in \mathcal{O}(U_j) \) such that \( (3.2) \) holds. Since \( h_1 \) is holomorphic on the bidisc \( \{|z_1| < 1\} \times \{|z_2| < \frac{1}{2}\} \), it has a Taylor expansion of the form \( h_1(z_1,z_2) = \sum_{m,n=0} a_{mn} z_1^m z_2^n \). And \( h_2 \) is holomorphic in the product \( \{\frac{1}{2} < |z_1| < 1\} \times \{|z_2| < 1\} \) of an annulus and a disc, so it has a Laurent expansion of the form \( h_2(z_1,z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=0} b_{mn} z_1^m z_2^n \). And the Laurent expansion of \( \frac{1}{z_1(z_2-a)} \) on the product \( U_1 \cap U_2 = \{\frac{1}{2} < |z_1| < 1\} \times \{|z_2| < 1\} \) is \( \sum_{\nu=0}^{\infty} a^{-(\nu+1)} z_1^{1-\nu} z_2^\nu \). Therefore from \( (3.2) \) and the uniqueness of Laurent expansions we see that we must have on \( U_1 \cap U_2 \), \( h_1 = 0 \), and \( h_2(z_1,z_2) = -\frac{1}{z_1(z_2-a)} \). But then \( h_2 \) does not extend to \( U_2 \) holomorphically because of the pole along \( z_2 = a \). By the uniqueness of analytic continuation, there can be no holomorphic \( h_j \in \mathcal{O}(U_j) \) satisfying \( (3.2) \).

(3) Canonical representation of domains in \( \mathbb{C}^n \): Let \( \Omega \in \mathbb{C}^n \) and \( \partial \Omega \) is \( C^2 \).

(a) Near any point \( z^0 \in \partial \Omega \), we can introduce holomorphic coordinates \( (z_1, \ldots, z_n) \) (the change of variables are given by holomorphic functions) centered at \( z^0 \) so that

\[
\Omega = \{ \text{Im}(z_n) > \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + E(z) \} 
\]

(3.5)

Here the \( \lambda_j \) are real numbers, \( z_j = x_j + iy_j \), and \( E(z) = x_n I(z') + Dx_n^2 + O(|z'|^2) \), as \( z \to 0 \). Also \( I(z') \) is a linear function of \( x_1, \ldots, x_{n-1}, y_1, \ldots, y_{n-1} \) and \( D \) is a real number.

(b) With an additional change of variables, the above representation can be simplified to state

\[
y_n = \sum_{j=1}^{n-1} \lambda_j |z_j|^2 + O(|z'|^2) 
\]

for \( z' \to 0 \).