11. The \(d\)-Neumann problem

To better understand the \(\bar{\partial}\)-Neumann problem, consider the Neumann problem for the DeRham \(d\)-complex on a domain in \(\mathbb{R}^N\). Details are given for comparison.

11.1. Spaces of forms.

**Definition 11.1.** For \(0 \leq r \leq N\) a tuple \(I = (i_1, \ldots, i_r), i_k \in \{1, \ldots, N\}\), is called a form summation multi-index \(I\). Denote by \(|I|\) the length \(r\) of \(I\) and write \(dx^I := dx_{i_1} \wedge \ldots \wedge dx_{i_r}\) and \(a^I := a_{i_1 \ldots i_r}\).

An \(r\)-form \(a\) on \(\Omega \subset \mathbb{R}^N\) with smooth boundary is a formal sum

\[ a := \sum_{|I|=r} a^I dx^I = \sum' a^I dx^I, \]

where \(a^I\) are functions on \(\bar{\Omega}\) and \(\sum'\) indicates summation over multi-indices \(I = (i_1, \ldots, i_r)\) with \(i_1 < \ldots < i_r\).

For \(u = \sum'_{|I|=p} u^I dx^I\) and \(v = \sum'_{|I|=p} v^I dx^I\), \(p\)-forms on \(\Omega\), define the scalar product on \(p\)-forms by

\[ (u, v)_{L^2(\Omega)} := \sum'_{|I|=p} \int_{\Omega} u^I v^I dV. \]  

(11.2)

Note only products of coefficients of \(u\) and \(v\) in the integral occur if the indices coincide. In particular, forms of different degree are orthogonal.

The main spaces of forms are listed for easy reference:

\[ L^2_r(\Omega) := \left\{ u = \sum'_{|I|=r} u^I dx^I : (u, u)_{L^2(\Omega)} < \infty \right\} \]
\[ \Lambda^r(\Omega) := \left\{ u = \sum'_{|I|=r} u^I dx^I : u^I \in C^\infty(\Omega) \right\} \]
\[ \Lambda^r_0(\Omega) := \left\{ u = \sum'_{|I|=r} u^I dx^I : u^I \in C^\infty_0(\Omega) \right\} \]
\[ \Lambda^r(\bar{\Omega}) := \left\{ u = \sum'_{|I|=r} u^I dx^I : u^I \in C^\infty(\bar{\Omega}) \right\} \]

11.2. Exterior differentiation. Use subscripts to denote derivatives: if \(f \in C^\infty(\Omega)\), let \(f_{x_k} = \frac{\partial f}{\partial x_k}\) for \(k \in \{1, 2, \ldots, N\}\).
Definition 11.3. For $u = \sum_{|I|=r} u_I dx^I \in \Lambda^r(\Omega)$ define the exterior derivative $d$ of $u$ by

$$du := \sum_{|I|=r} \sum_{k=1}^{N} (u_I)_{x_k} dx_k \wedge dx^I.$$ 

Denote by $\sigma_{KI}^J$ the sign of the permutation that takes the $r+1$-tuple $(K, I) = (K, i_1, \ldots, i_r)$, where $i_1 < \ldots < i_r$, to $J := (j_1, \ldots, j_{r+1})$, where $j_1 < \ldots < j_{r+1}$, if $\{KI\} = \{J\}$ as sets and 0 otherwise. Then

$$du = \sum_{|J|=r+1} \sigma_{KI}^J (u_I)_{x_k} dx^J.$$ 

Lifting $d$ to higher-order forms by linearity, $d : \Lambda^r(\Omega) \to \Lambda^{r+1}(\Omega)$, gives the de Rham complex

$$\Lambda^0(\Omega) \overset{d}{\rightarrow} \Lambda^1(\Omega) \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} \Lambda^N(\Omega) \overset{d}{\rightarrow} 0.$$ 

Note the crucial condition $d \circ d \equiv 0$.

Extend this complex to the $L^2$-setting. At each form level let the extension, still denoted by $d$, be defined on $\text{Dom}(d) := \{ u \in L^2_q(\Omega) : \text{each component of } du \text{ is in } L^2_{q+1}(\Omega) \}$, where $d$ acts on $L^2_q(\Omega)$ in the distributional sense. It can be shown that $\text{Dom}(d)$ is dense in $L^2_q(\Omega)$. The pair $(d, \text{Dom}(d))$ is the maximal $L^2$-extension of $d$ and the following complex is obtained

$$L^2_0(\Omega) \overset{d}{\rightarrow} L^2_1(\Omega) \overset{d}{\rightarrow} \ldots \overset{d}{\rightarrow} L^2_N(\Omega).$$

11.3. The adjoint of $d$.

Definition 11.4. The formal adjoint $\delta$ of $d$ must satisfy

$$(du, v)_{L^2(\Omega)} = (u, \delta v)_{L^2(\Omega)} \quad \forall u \in \Lambda^r(\Omega), v \in \Lambda^{r+1}(\Omega).$$

In case $u, v$ are compactly supported, the formal adjoint is obtained by integration by parts. For $r = 0$ the computation follows. Let $v = \sum_{j=1}^{N} v_j dx_j \in \Lambda^1_0(\Omega)$ and $u \in \Lambda^0_0(\Omega) = C_0^\infty(\Omega)$, then

$$(du, v) = \left( \sum_{j=1}^{N} u_{x_j} dx_j, \sum_{k=1}^{N} v_k dx_k \right) = \int_{\Omega} \sum_{j=1}^{N} u_{x_j} v_j dV$$

$$\overset{\text{IBP}}{=} -\int_{\Omega} \sum_{j=1}^{N} u(v_j)_{x_j} dV$$

$$= -\int_{\Omega} u \left( \sum_{j=1}^{N} (v_j)_{x_j} \right) dV =: (u, \delta v).$$

Hence $\delta$ is the divergence operator. Now drop the compact support assumption and repeat this computation for $u \in \Lambda^0(\Omega)$ and $v \in \Lambda^1(\Omega)$. Let $\rho$ be a defining function for $\Omega$ with
$|\nabla \rho| = 1$ on $b\Omega$. Then

$$ (du, v) = (u, \delta v) + \int_{b\Omega} u \sum_{j=1}^{N} v_j \rho x_j dS. \quad (11.5) $$

Thus $(du, v) = (u, \delta v)$ whenever $\sum_{j=1}^{N} v_j \rho x_j = 0$ on $b\Omega$.

The $L^2$ adjoint of $d$, denoted by $d^*$, is defined on

$$ \text{Dom}(d^*) := \left\{ u \in L^2_q(\Omega) : \exists C > 0, \text{ such that } |(u, dv)_{L^2_q(\Omega)}| \leq C\|v\|_{L^2_q(\Omega)}, \forall v \in \text{Dom}(d) \right\}. $$

Here $\|\cdot\|_{L^2_q(\Omega)}$ is the norm induced by the scalar product in the sense of (11.2). The following calculation guarantees the existence of $d^* u$ for $u \in \text{Dom}(d^*)$.

**Theorem 11.6.** If $u \in \text{Dom}(d^*)$, there exists a unique element $d^* u$ in $L^2_{q-1}(\Omega)$ satisfying

$$ (u, dv)_{L^2_q(\Omega)} = (d^* u, v)_{L^2_{q-1}(\Omega)} \quad \forall v \in \text{Dom}(d). $$

Moreover, $d^*$ is a linear Hilbert space operator.

**Proof.** For $u \in \text{Dom}(d^*)$ fixed, consider the linear functional

$$ L : \text{Dom}(d) \subset L^2_{q-1}(\Omega) \to \mathbb{R} $$

$$ v \mapsto (u, dv)_{L^2_q(\Omega)}. $$

By the definition of $\text{Dom}(d^*)$, $L$ is bounded. Extend $L$ to $L^2_{q-1}(\Omega)$ by continuity. The Riesz representation theorem gives a unique $w \in L^2_{q-1}(\Omega)$ such that $(w, v) = (u, dv)$ holds. Set $d^* u := w$.

To see that $d^*$ is linear, let $u_1, u_2 \in L^2_q(\Omega)$ and $v \in L^2_{q-1}(\Omega)$. Then

$$ (d^* u_1 + d^* u_2, v) = (d^* u_1, v) + (d^* u_2, v) = (u_1, dv) + (u_2, dv) = (u_1 + u_2, dv) = (d^* (u_1 + u_2), v), $$

i.e., $d^* u_1 + d^* u_2 = d^* (u_1 + u_2)$. A similar calculation yields $d^* \lambda u = \lambda d^* u$ for $\lambda \in \mathbb{R}$. Since $\text{Dom}(d)$ is dense in $L^2_q(\Omega)$, $(d^*, \text{Dom}(d^*))$ is a Hilbert space operator. \qed

For the rest of this section, only consider the case $q = 1$.

**Definition 11.7.**

$$ D^1(\Omega) := \text{Dom}(d^*) \cap \Lambda^1(\Omega) $$

**Proposition 11.8.** Let $u = \sum_{j=1}^{N} u_j dx_j$ be a 1-form with $u \in \Lambda^1(\Omega)$. Then $u \in \text{Dom}(d^*)$ if and only if $u$ satisfies the following boundary condition for a defining function $r$ for $\Omega$ with $|\nabla r| = 1$ on $b\Omega$:

$$ \sum_{j=1}^{N} r_{x_j}(x) u_j(x) = 0 \quad \forall x \in b\Omega. $$
Thus for any $v$ respect to the graph norm $\|v\|$. In particular, choose $\tilde{f} := \sum_{j=1}^{N} r_{x_j} u_j \in \Lambda^0(\Omega)$. It follows that $\sum_{j=1}^{N} r_{x_j} u_j = 0$ on $b\Omega.

Conversely, if $\sum_{j=1}^{N} r_{x_j}(x) u_j(x) = 0$ on $b\Omega$, then (11.5) gives $d^* u = \delta u$ uniquely exists. Thus for any $v \in L^2(\Omega)$ we have

$$|(u, dv)_{L^2(\Omega)}| = |(\delta u, v)| \leq \|\delta u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)},$$

i.e., $u \in \text{Dom}(d^*)$. \qed

Hörmander [?], Proposition 2.1.1, proved, that $\mathcal{D}^1$ is dense in $\text{Dom}(d) \cap \text{Dom}(d^*)$ with respect to the graph norm $\|u\|_{L^2} + \|du\|_{L^2} + \|d^* u\|_{L^2}$.

The following identity is the basic fact.

**Proposition 11.9.** Let $r$ be a defining function for $\Omega$ with $|\nabla r| = 1$ on $b\Omega$. If $u \in \mathcal{D}^1$, then

$$\|du\|_{L^2(\Omega)}^2 + \|d^* u\|_{L^2(\Omega)}^2 = \sum_{j,k=1}^{N} \|(u_k)_{x_j} - (u_j)_{x_k}\|^2_{L^2(\Omega)} + \int_{b\Omega} \sum_{j,k=1}^{N} r_{x_j x_k} u_j u_k dS.$$

**Proof.** Let $u = \sum_{j=1}^{N} u_j x_j \in \mathcal{D}^1$. Then

$$du = \sum_{j<k} \left( (u_k)_{x_j} - (u_j)_{x_k} \right) dx_j \wedge dx_k.$$

Thus

$$\|du\|_{L^2}^2 = \sum_{j<k} \left\| (u_k)_{x_j} - (u_j)_{x_k} \right\|^2_{L^2} = \sum_{j<k} \int_{\Omega} \left| (u_k)_{x_j} - (u_j)_{x_k} \right|^2 d\Omega$$

$$= \sum_{j,k=1}^{N} \left\| (u_k)_{x_j} \right\|^2_{L^2} - \sum_{j,k=1}^{N} \left( (u_k)_{x_j}, (u_j)_{x_k} \right).$$

An inspection of $M$ shows that

$$M = \sum_{j,k=1}^{N} \left( (u_j)_{x_k}, (u_k)_{x_j} \right) \text{ IBP } \sum_{j,k=1}^{N} \left[ - \left( \frac{\partial}{\partial x_j} ((u_j)_{x_k}), u_k \right) + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right]$$

$$= \sum_{j,k=1}^{N} \left[ - \left( \frac{\partial}{\partial x_k} ((u_j)_{x_j}), u_k \right) + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right],$$

since $\left[ \frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right] = 0$ holds. A further integration by parts gives

$$M = \sum_{j,k=1}^{N} \left[ \left( (u_j)_{x_j}, (u_k)_{x_k} \right) - \int_{b\Omega} r_{x_k} (u_j)_{x_j} u_k dS + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right].$$
Since \( u \in \mathcal{D}^1 \), \( d^* u = \delta u = -\sum_{j=1}^{N} \frac{\partial u_j}{\partial x_j} \). Hence

\[
\|d^* u\|_{L^2}^2 = \int_{\Omega} \left( \sum_{j=1}^{N} (u_j x_j) \right)^2 = \sum_{j,k=1}^{N} (u_j x_j)(u_k x_k) dV.
\]

Thus

\[
M = \|d^* u\|_{L^2}^2 - \int_{b\Omega} \sum_{j,k=1}^{N} r_{x_j} (u_j x_j) u_k dS + \int_{b\Omega} \sum_{j,k=1}^{N} r_{x_j} (u_j x_k) u_k dS.
\]

Putting these computations together,

\[
\|du\|_{L^2}^2 = \sum_{j,k=1}^{N} \|(u_k x_j)\|_{L^2}^2 - \|d^* u\|_{L^2}^2 + bI_1 - bI_2.
\]

The term \( bI_1 \) directly vanishes, since \( u \in \mathcal{D}^1 \) implies

\[
bI_1 = \int_{b\Omega} \left( \sum_{j=1}^{N} (u_j x_j) \left( \sum_{k=1}^{N} r_{x_k} u_k \right) \right) dS = 0.
\]

Since \( \sum_{k=1}^{N} r_{x_k} u_k = 0 \) on \( b\Omega \) holds, the vector field given by \( X := \sum_{k=1}^{N} u_k \frac{\partial}{\partial x_k} \) is tangent to \( b\Omega \), i.e., \( X(r) = 0 \) on \( b\Omega \). Thus \( X(g) = 0 \) for any \( g \) that satisfies \( g = 0 \) on \( b\Omega \). In particular, taking \( g = \sum_{j=1}^{N} r_{x_j} u_j \) yields

\[
0 = X(g) = \sum_{k=1}^{N} u_k \frac{\partial}{\partial x_k} \left( \sum_{j=1}^{N} r_{x_j} u_j \right) = \sum_{j,k=1}^{N} u_k r_{x_j x_k} u_j + u_k r_{x_j} (u_j x_k).
\]

Substituting this into \( bI_2 \) gives the formula claimed. \( \square \)

The **d-Neumann problem** on 1-forms can now be stated:
Given \( \Omega \) smoothly bounded in \( \mathbb{R}^N \) and \( f \in L^2_1(\Omega) \) with \( f = \sum_{j=1}^{N} f_j dx_j \), find \( u \in L^2_1(\Omega) \) with \( u = \sum_{j=1}^{N} u_j dx_j \), such that

\[
\begin{cases}
\Delta_1 u := (dd^* + d^* d)u = f, \\
u \in \text{Dom}(d^*), \\
du \in \text{Dom}(d^*).
\end{cases}
\]

Abbreviate the last two conditions by the single inclusion \( u \in \text{Dom}(\Delta_1) \).

This is an example of a boundary value problem. There are various regularity issues that can be addressed but initially it suffices to find a bounded, linear operator on \( L^2(\Omega) \) solving this system of \( N \) equations. Thus we seek an operator, the so called **d-Neumann operator** \( N^d \), with \( N^d : L^2_1(\Omega) \to L^2_1(\Omega) \), such that

\[
\begin{cases}
\Delta_1(N^d f) = f, \\
N^d f \in \text{Dom}(d^*), \\
d(N^d f) \in \text{Dom}(d^*).
\end{cases}
\]
Theorem 11.10. If \( \Omega \subset \mathbb{R}^N \) is a smoothly bounded domain, then
\[
\|u\|_1^2 \lesssim \|du\|^2_{L^2} + \|d^*u\|^2_{L^2} + \|u\|^2_{L^2}, \ u \in \text{Dom}(\Delta_1) \tag{11.11}
\]
where \(\| \cdot \|_1\) is the Sobolev norm on \(W^1(\Omega)\). (See Supplemental lecture 1.5, (1.2).)

\[
\|du\|^2_{L^2} + \|d^*u\|^2_{L^2} + \|u\|^2_{L^2} \geq \sum_{j,k=1}^{N} \|(u_j)_{x_j}\|^2_{L^2} - \left| \int_{\partial \Omega} \sum_{j,k=1}^{N} r_{x_jx_k} u_j u_k dS \right| + \|u\|^2_{L^2}.
\]
Choose a positive constant \(M\) such that \(M > \sup_{j,k \in \{1,...,N\}} \sup_{x \in \partial \Omega} \{ |r_{x_jx_k}(x)| \}\). Then
\[
\|du\|^2_{L^2} + \|d^*u\|^2_{L^2} + \|u\|^2_{L^2} \geq \sum_{j,k=1}^{N} \|(u_j)_{x_j}\|^2_{L^2} + \|u\|^2_{L^2} - M \int_{\partial \Omega} |u|^2 dS.
\]
We claim that the term \(\int_{\partial \Omega} |u|^2 dS\) can be absorbed by \(\|u\|^2_D\) and \(\|u\|^2_{L^2}\). Applying Stokes Theorem, followed by the Cauchy–Schwarz inequality, yields
\[
\int_{\partial \Omega} |u|^2 dS = \int_{\Omega} d(|u|^2) dV = 2 \int_{\Omega} du \wedge u \\
\leq 2 \left( \int_{\Omega} |du|^2 \right)^{1/2} \left( \int_{\Omega} |u|^2 \right)^{1/2}.
\]
It follows from the (sc)–(lc) inequality that
\[
\int_{\partial \Omega} |u|^2 dS \leq \varepsilon \|du\|^2_{L^2} + \frac{1}{\varepsilon} \|u\|^2_{L^2} = \varepsilon \|u\|^2_D + \frac{1}{\varepsilon} \|u\|^2_{L^2}
\]
for any \(\varepsilon > 0\). Thus there is a constant \(C\) such that (11.11) holds, as claimed. \(\Box\)

As a first consequence, it follows that \(N^d\) exists for the operator \(\Delta_1 + id\), where \(id\) is the identity on \(L^2_1\).

However it also follows that the space of harmonic form \(\mathcal{H}^1(\Omega) = \{ u \in \text{Dom}(\Delta_1) : \Delta_1 u = 0 \}\)
is a finite dimensional subspace of \(L^2_1\). First a general fact

Lemma 11.12. Let \(S\) be a linear subspace of \(W^1_0(\Omega)\). If there exists a constant \(C\) such that
\[
\|g\|_1 \leq C \|g\|_{L^2(\Omega)} \quad \forall g \in S, \tag{11.13}
\]
then \(\dim(S) < \infty\).

Proof. It follows from the Bolzano–Weierstraß theorem that the normed subspace \(S \subset W^1_0(\Omega)\) is finite dimensional if every bounded sequence in \(S\) has a convergent subsequence in \(S\).
Let \( \{g_n\} \) be a bounded sequence in \( S \subset W^1_0(\Omega) \). The Rellich–Kondrachov theorem implies that there is a convergent subsequence \( \{g_{n_k}\} \) in \( L^2(\Omega) \); let \( g := \lim_{k \to \infty} g_{n_k} \). Inequality (11.13) says that

\[
\|g_{n_k} - g_{n_j}\|_1 \leq C\|g_{n_k} - g_{n_j}\|_{L^2(\Omega)}.
\]

Thus \( g \) belongs to \( W^1_0(\Omega) \). It remains to show \( g \in S \):

\[
\|g\|_1 \leq \|g - g_{n_k}\|_1 + \|g_{n_k}\|_1 \leq \varepsilon + C\|g_{n_k}\|_{L^2(\Omega)} \\
\leq \varepsilon + C\|g - g_{n_k}\|_{L^2(\Omega)} + C\|g\|_{L^2(\Omega)} \\
\leq \varepsilon(C + 1) + C\|g\|_{L^2(\Omega)}.
\]

\[\square\]

**Corollary 11.14.** If \( \Omega \subset \mathbb{R}^N \) is a smoothly bounded domain, then \( \dim \mathcal{H}^1(\Omega) < \infty \).

**Proof.** Integrating by parts gives, for \( u \in \text{Dom}(\Delta_1) \),

\[
(\Delta_1 u, u) + \|u\|_{L^2}^2 = (du, du) + (d^*u, d^*u) + \|u\|_{L^2}^2 \\
= \|du\|_{L^2}^2 + ||d^*u||_{L^2}^2 + \|u\|_{L^2}^2 \\
\geq \|u\|_1^2.
\]

If \( \Delta_1 u = 0 \), it follows that \( \|u\|_{L^2}^2 \geq \|u\|_1^2 \). The previous Lemma finishes the proof. \[\square\]