MSRI2018, SUPPLEMENTAL LECTURE 2

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READ WITH CAUTION, TYPOS ABOUND.

11. The d-Neumann problem

To better understand the $\bar{\partial}$-Neumann problem, consider the Neumann problem for the DeRham d-complex on a domain in $\mathbb{R}^N$. Details are given for comparison.

11.1. Spaces of forms.

Definition 11.1. *For $0 \le r \le N$ a tuple $I = (i_1, \ldots, i_r)$, $i_k \in \{1, \ldots, N\}$, is called a **form summation multi-index I**. Denote by $|I|$ the length r of I and write $dx^I := dx_{i_1} \wedge \ldots \wedge dx_{i_r}$ and $a_I := a_{i_1 \cdots i_r}$.*

*An **r-form a** on $\Omega \subset\subset \mathbb{R}^N$ with smooth boundary is a formal sum*

$$a := \sum\nolimits_{|I|=r} a_I dx^I = \sum\nolimits'_{|I|=r} \tilde{a}_I dx^I,$$

where a_I are functions on $\bar{\Omega}$ and $\sum'$ indicates summation over multi-indices $I = (i_1, \ldots, i_r)$ with $i_1 < \ldots < i_r$.

For $u = \sum'_{|I|=p} u_I dx^I$ and $v = \sum'_{|I|=p} v_I dx^I$, p-forms on Ω, define the scalar product on p-forms by

$$(u, v)_{L^2(\Omega)} := \sum\nolimits'_{|I|=p} \int_\Omega u_I v_I dV. \tag{11.2}$$

Note only products of coefficients of u and v in the integral occur if the indices coincide. In particular, forms of different degree are orthogonal.

The main spaces of forms are listed for easy reference:

$$L^2_r(\Omega) := \left\{ u = \sum\nolimits'_{|I|=r} u_I dx^I : (u, u)_{L_2(\Omega)} < \infty \right\}$$
$$\Lambda^r(\Omega) := \left\{ u = \sum\nolimits'_{|I|=r} u_I dx^I : u_I \in \mathcal{C}^\infty(\Omega) \right\}$$
$$\Lambda^r_0(\Omega) := \left\{ u = \sum\nolimits'_{|I|=r} u_I dx^I : u_I \in \mathcal{C}^\infty_0(\Omega) \right\}$$
$$\Lambda^r(\bar{\Omega}) := \left\{ u = \sum\nolimits'_{|I|=r} u_I dx^I : u_I \in \mathcal{C}^\infty(\bar{\Omega}) \right\}$$

11.2. Exterior differentiation.

Use subscripts to denote derivatives: if $f \in \mathcal{C}^\infty(\Omega)$, let $f_{x_k} = \frac{\partial f}{\partial x_k}$ for $k \in \{1, 2, \ldots, N\}$.

Definition 11.3. *For $u = \sum'_{|I|=r} u_I dx^I \in \Lambda^r(\Omega)$ define the* ***exterior derivative d of u*** *by*

$$du := \sum\nolimits'_{|I|=r} \sum_{k=1}^{N} (u_I)_{x_k} dx_k \wedge dx^I.$$

Denote by σ^J_{KI} the sign of the permutation that takes the $r+1$-tuple $(K, I) = (K, i_1, \ldots, i_r)$, where $i_1 < \ldots < i_r$, to $J := (j_1, \ldots, j_{r+1})$, where $j_1 < \ldots < j_{r+1}$, if $\{KI\} = \{J\}$ as sets and 0 otherwise. Then

$$du = \sum\nolimits'_{|J|=r+1} \sigma^J_{kI} (u_I)_{x_k} dx^J.$$

Lifting d to higher-order forms by linearity, $d : \Lambda^r(\Omega) \to \Lambda^{r+1}(\Omega)$, gives the **de Rham complex**

$$\Lambda^0(\Omega) \xrightarrow{d} \Lambda^1(\Omega) \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^N(\Omega) \xrightarrow{d} 0.$$

Note the crucial condition $d \circ d \equiv 0$.

Extend this complex to the L^2-setting. At each form level let the extension, still denoted by d, be defined on

$$\mathrm{Dom}(d) := \left\{ u \in L^2_q(\Omega) : \text{ each component of } du \text{ is in } L^2_{q+1}(\Omega) \right\},$$

where d acts on $L^2_q(\Omega)$ in the distributional sense. It can be shown that

$$\mathrm{Dom}(d) = \left\{ u \in L^2_q(\Omega) : \ \exists \{v_n\} \subset \Lambda^q(\bar{\Omega}) : \ v_n \to u \text{ in } L^2(\Omega), \ \{dv_n\} \text{ a Cauchy sequence in } L^2_{q+1}(\Omega) \right\}$$

and $\mathrm{Dom}(d)$ is dense in $L^q_2(\Omega)$. The pair $(d, \mathrm{Dom}(d))$ is the **maximal L_2-extension of d** and the following complex is obtained

$$L^2_0(\Omega) \underset{d^*}{\overset{d}{\rightleftarrows}} L^2_1(\Omega) \underset{d^*}{\overset{d}{\rightleftarrows}} \ldots \underset{d^*}{\overset{d}{\rightleftarrows}} L^2_N(\Omega).$$

11.3. **The adjoint of d.**

Definition 11.4. *The* ***formal adjoint δ of d*** *must satisfy*

$$(du, v)_{L^2(\Omega)} = (u, \delta v)_{L^2(\Omega)} \qquad \forall u \in \Lambda^r(\Omega), v \in \Lambda^{r+1}(\Omega).$$

In case u, v are compactly supported, the formal adjoint is obtained by integration by parts. For $r = 0$ the computation follows. Let $v = \sum_{j=1}^N v_j dx_j \in \Lambda^1_0(\Omega)$ and $u \in \Lambda^0_0(\Omega) = \mathcal{C}^\infty_0(\Omega)$, then

$$\begin{aligned}
(du, v) &= \Big(\sum_{j=1}^N u_{x_j} dx_j, \sum_{k=1}^N v_k dx_k \Big) = \int_\Omega \sum_{j=1}^N u_{x_j} v_j dV \\
&\overset{\text{IBP}}{=} - \int_\Omega \sum_{j=1}^N u (v_j)_{x_j} dV \\
&= - \int_\Omega u \Big(\sum_{j=1}^N (v_j)_{x_j} \Big) dV =: (u, \delta v).
\end{aligned}$$

Hence δ is the divergence operator. Now drop the compact support assumption and repeat this computation for $u \in \Lambda^0(\Omega)$ and $v \in \Lambda^1(\Omega)$. Let ρ be a defining function for Ω with

$|\nabla\rho| = 1$ on $b\Omega$. Then

$$(du, v) = (u, \delta v) + \int_{b\Omega} u \sum_{j=1}^{N} v_j \rho_{x_j} dS. \tag{11.5}$$

Thus $(du, v) = (u, \delta v)$ whenever $\sum_{j=1}^{N} v_j \rho_{x_j} = 0$ on $b\Omega$.

The L^2 **adjoint of** $\boldsymbol{d}$, denoted by $\boldsymbol{d^*}$, is defined on

$$\mathrm{Dom}(d^*) := \left\{ u \in L^2_q(\Omega) : \ \exists C > 0, \text{ such that } |(u, dv)_{L^2_q(\Omega)}| \leq C\|v\|_{L^2_{q-1}(\Omega)}, \ \forall v \in \mathrm{Dom}(d) \right\}.$$

Here $\|.\|_{L^2_q(\Omega)}$ is the norm induced by the scalar product in the sense of (11.2). The following calculation guarantees the existence of d^*u for $u \in \mathrm{Dom}(d^*)$.

Theorem 11.6. *If $u \in Dom(d^*)$, there exists a unique element d^*u in $L^2_{q-1}(\Omega)$ satisfying*

$$(u, dv)_{L^2_q(\Omega)} = (d^*u, v)_{L^2_{q-1}(\Omega)} \qquad \forall v \in Dom(d).$$

Moreover, d^* is a linear Hilbert space operator.

Proof. For $u \in \mathrm{Dom}(d^*)$ fixed, consider the linear functional

$$\begin{aligned} L : \mathrm{Dom}(d) \subset L^2_{q-1}(\Omega) &\to \mathbb{R} \\ v &\mapsto (u, dv)_{L^2_q(\Omega)}. \end{aligned}$$

By the definition of $\mathrm{Dom}(d^*)$, L is bounded. Extend L to $L^2_{q-1}(\Omega)$ by continuity. The Riesz representation theorem gives a unique $w \in L^2_{q-1}(\Omega)$ such that $(w, v) = (u, dv)$ holds. Set $d^*u := w$.

To see that d^* is linear, let $u_1, u_2 \in L^2_q(\Omega)$ and $v \in L^2_{q-1}(\Omega)$. Then

$$\begin{aligned} (d^*u_1 + d^*u_2, v) &= (d^*u_1, v) + (d^*u_2, v) = (u_1, dv) + (u_2, dv) \\ &= (u_1 + u_2, dv) = (d^*(u_1 + u_2), v), \end{aligned}$$

i.e., $d^*u_1 + d^*u_2 = d^*(u_1 + u_2)$. A similar calculation yields $d^*\lambda u = \lambda d^*u$ for $\lambda \in \mathbb{R}$. Since $\mathrm{Dom}(d)$ is dense in $L^2_q(\Omega)$, $(d^*, \mathrm{Dom}(d^*))$ is a Hilbert space operator. □

For the rest of this section, only consider the case $q = 1$.

Definition 11.7.

$$\mathcal{D}^1(\Omega) := Dom(d^*) \cap \Lambda^1(\bar{\Omega})$$

Proposition 11.8. *Let $u = \sum_{j=1}^{N} u_j dx_j$ be a 1-form with $u \in \Lambda^1(\bar{\Omega})$.*
Then $u \in Dom(d^*)$ if and only if u satisfies the following boundary condition for a defining function r for Ω with $|\nabla r| = 1$ on $b\Omega$:

$$\sum_{j=1}^{N} r_{x_j}(x) u_j(x) = 0 \qquad \forall x \in b\Omega.$$

Proof. Let $u \in \Lambda^1(\bar{\Omega})$ and suppose $u \in \mathrm{Dom}(d^*)$. Then by (11.5),

$$\int_{b\Omega} \sum_{j=1}^N r_{x_j} u_j f = 0 \qquad \forall f \in \mathrm{Dom}(d).$$

In particular, choose $\tilde{f} := \sum_{j=1}^N r_{x_j} u_j \in \Lambda^0(\bar{\Omega})$. It follows that $\sum_{j=1}^N r_{x_j} u_j = 0$ on $b\Omega$.

Conversely, if $\sum_{j=1}^N r_{x_j}(x) u_j(x) = 0$ on $b\Omega$, then (11.5) gives $d^* u = \delta u$ uniquely exists. Thus for any $v \in L^2(\Omega)$ we have

$$|(u, dv)_{L^2(\Omega)}| = |(\delta u, v)| \leq \|\delta u\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)},$$

i.e., $u \in \mathrm{Dom}(d^*)$. □

Hörmander [**?**], Proposition 2.1.1, proved, that $\mathcal{D}^1$ is dense in $\mathrm{Dom}(d) \cap \mathrm{Dom}(d^*)$ with respect to the graph norm $\|u\|_{L^2} + \|du\|_{L^2} + \|d^* u\|_{L^2}$.

The following identity is the basic fact.

Proposition 11.9. *Let r be a defining function for Ω with $|\nabla r| = 1$ on $b\Omega$. If $u \in \mathcal{D}^1$, then*

$$\|du\|^2_{L^2(\Omega)} + \|d^* u\|^2_{L^2(\Omega)} = \sum_{j,k=1}^N \|(u_k)_{x_j}\|^2_{L^2(\Omega)} + \int_{b\Omega} \sum_{j,k=1}^N r_{x_j x_k} u_j u_k dS.$$

Proof. Let $u = \sum_{j=1}^N u_j dx_j \in \mathcal{D}^1$. Then

$$du = \sum_{j<k} \Big((u_k)_{x_j} - (u_j)_{x_k} \Big) dx_j \wedge dx_k.$$

Thus

$$\begin{aligned} \|du\|^2_{L^2} &= \sum_{j<k} \Big\| (u_k)_{x_j} - (u_j)_{x_k} \Big\|^2_{L^2} = \sum_{j<k} \int_\Omega \Big| (u_k)_{x_j} - (u_j)_{x_k} \Big|^2 dV \\ &= \sum_{j,k=1}^N \Big\| (u_k)_{x_j} \Big\|^2_{L^2} - \underbrace{\sum_{j,k=1}^N \Big((u_k)_{x_j}, (u_j)_{x_k} \Big)}_{=:M}. \end{aligned}$$

An inspection of M shows that

$$\begin{aligned} M &= \sum_{j,k=1}^N \Big((u_j)_{x_k}, (u_k)_{x_j} \Big) \overset{\text{IBP}}{=} \sum_{j,k=1}^N \left[- \left(\frac{\partial}{\partial x_j} \Big((u_j)_{x_k} \Big), u_k \right) + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right] \\ &= \sum_{j,k=1}^N \left[- \left(\frac{\partial}{\partial x_k} \Big((u_j)_{x_j} \Big), u_k \right) + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right], \end{aligned}$$

since $\left[\frac{\partial}{\partial x_k}, \frac{\partial}{\partial x_j} \right] = 0$ holds. A further integration by parts gives

$$M = \sum_{j,k=1}^N \left[\Big((u_j)_{x_j}, (u_k)_{x_k} \Big) - \int_{b\Omega} r_{x_k} (u_j)_{x_j} u_k dS + \int_{b\Omega} r_{x_j} (u_j)_{x_k} u_k dS \right].$$

Since $u \in \mathcal{D}^1$, $d^*u = \delta u = -\sum_{j=1}^N \frac{\partial u_j}{\partial x_j}$. Hence

$$\|d^*u\|_{L^2}^2 = \int_\Omega \Big|\sum_{j=1}^N (u_j)_{x_j}\Big|^2 = \sum_{j,k=1}^N \int_\Omega (u_j)_{x_j}(u_k)_{x_k} dV.$$

Thus

$$M = \|d^*u\|_{L^2}^2 - \underbrace{\int_{b\Omega} \sum_{j,k=1}^N r_{x_k}(u_j)_{x_j} u_k dS}_{=:bI_1} + \underbrace{\int_{b\Omega} \sum_{j,k=1}^N r_{x_j}(u_j)_{x_k} u_k dS}_{=:bI_2}.$$

Putting these computations together,

$$\|du\|_{L^2}^2 = \sum_{j,k=1}^N \|(u_k)_{x_j}\|_{L^2}^2 - \|d^*u\|_{L^2}^2 + bI_1 - bI_2.$$

The term bI_1 directly vanishes, since $u \in \mathcal{D}^1$ implies

$$bI_1 = \int_{b\Omega} \left(\sum_j^N (u_j)_{x_j} \underbrace{\Big(\sum_{k=1}^N r_{x_k} u_k\Big)}_{=0} \right) dS = 0.$$

Since $\sum_{k=1}^N r_{x_k} u_k = 0$ on $b\Omega$ holds, the vector field given by $X := \sum_{k=1}^N u_k \frac{\partial}{\partial x_k}$ is tangent to $b\Omega$, i.e., $X(r) = 0$ on $b\Omega$. Thus $X(g) = 0$ for any g that satisfies $g = 0$ on $b\Omega$. In particular, taking $g = \sum_{j=1}^N r_{x_j} u_j$ yields

$$0 = X(g) = \sum_{k=1}^N u_k \frac{\partial}{\partial x_k}\Big(\sum_{j=1}^N r_{x_j} u_j\Big) = \sum_{j,k=1}^N \Big(u_k r_{x_j x_k} u_j + u_k r_{x_j}(u_j)_{x_k}\Big).$$

Substituting this into bI_2 gives the formula claimed. □

The **d-Neumann problem** on 1-forms can now be stated:
Given Ω smoothly bounded in $\mathbb{R}^N$ and $f \in L_1^2(\Omega)$ with $f = \sum_{j=1}^N f_j dx_j$, find $u \in L_1^2(\Omega)$ with $u = \sum_{j=1}^N u_j dx_j$, such that

$$\begin{cases} \Delta_1 u := (dd^* + d^*d)u = f, \\ \quad u \in \mathrm{Dom}(d^*), \\ \quad du \in \mathrm{Dom}(d^*). \end{cases}$$

Abbreviate the last two conditions by the single inclusion $u \in \mathrm{Dom}\,(\Delta_1)$.

This is an example of a boundary value problem. There are various regularity issues that can be addressed but initially it suffices to find a bounded, linear operator on $L^2(\Omega)$ solving this system of N equations. Thus we seek an operator, the so called **d-Neumann operator $\boldsymbol{N^d}$**, with $N^d : L_1^2(\Omega) \to L_1^2(\Omega)$, such that

$$\begin{cases} \Delta_1(N^d f) = f, \\ \quad N^d f \in \mathrm{Dom}(d^*), \\ \quad d(N^d f) \in \mathrm{Dom}(d^*). \end{cases}$$

Theorem 11.10. *If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain, then*

$$\|u\|_1^2 \lesssim \|du\|_{L^2}^2 + \|d^*u\|_{L^2}^2 + \|u\|_{L^2}^2, \quad u \in Dom\,(\Delta_1) \tag{11.11}$$

where $\|.\|_1$ is the Sobolev norm on $W^1(\Omega)$. (See Supplemental lecture 1.5, (1.2).)

Proof. Proposition 11.9 gives

$$\|du\|_{L^2}^2 + \|d^*u\|_{L^2}^2 + \|u\|_{L^2}^2 \geq \sum_{j,k=1}^N \|(u_k)_{x_j}\|_{L^2}^2 - \left| \int_{b\Omega} \sum_{j,k=1}^N r_{x_j x_k} u_j u_k dS \right| + \|u\|_{L^2}^2.$$

Choose a positive constant M such that $M > \sup_{j,k\in\{1,\dots,N\}} \sup_{x\in b\Omega}\{|r_{x_j x_k}(x)|\}$. Then

$$\|du\|_{L^2}^2 + \|d^*u\|_{L^2}^2 + \|u\|_{L^2}^2 \geq \underbrace{\sum_{j,k=1}^N \|(u_k)_{x_j}\|_{L^2}^2}_{=\|u\|_D^2} + \|u\|_{L^2}^2 - M\int_{b\Omega} |u|^2 dS.$$

We claim that the term $\int_{b\Omega} |u|^2 dS$ can be absorbed by $\|u\|_D^2$ and $\|u\|_{L^2}^2$. Applying Stokes Theorem, followed by the Cauchy–Schwarz inequality, yields

$$\begin{aligned}\int_{b\Omega} |u|^2 dS &= \int_\Omega d(|u|^2) dV = 2\int_\Omega du \wedge u \\ &\leq 2\Big(\int_\Omega |du|^2\Big)^{\frac{1}{2}} \Big(\int_\Omega |u|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

It follows from the (sc)–(lc) inequality that

$$\int_{b\Omega} |u|^2 dS \leq \varepsilon \|du\|_{L^2}^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2 = \varepsilon\|u\|_D^2 + \frac{1}{\varepsilon}\|u\|_{L^2}^2$$

for any $\varepsilon > 0$. Thus there is a constant C such that (11.11) holds, as claimed. □

As a first consequence, it follows that N^d exists for the operator $\Delta_1 + id$, where id is the identity on L_1^2.

However it also follows that the space of harmonic form

$$\mathcal{H}^1(\Omega) = \{u \in \mathrm{Dom}\,(\Delta_1) : \Delta_1 u = 0\}$$

is a finite dimensional subspace of L_1^2. First a general fact

Lemma 11.12. *Let S be a linear subspace of $W_0^1(\Omega)$. If there exists a constant C such that*

$$\|g\|_1 \leq C\|g\|_{L^2(\Omega)} \qquad \forall g \in S, \tag{11.13}$$

then $\dim(S) < \infty$.

Proof. It follows from the Bolzano–Weierstraß theorem that the normed subspace $S \subset W_0^1(\Omega)$ is finite dimensional if every bounded sequence in S has a convergent subsequence in S.

Let $\{g_n\}$ be a bounded sequence in $S \subset W_0^1(\Omega)$. The Rellich–Kondrachov theorem implies that there is a convergent subsequence $\{g_{n_k}\}$ in $L^2(\Omega)$; let $g := \lim_{k\to\infty} g_{n_k}$. Inequality (11.13) says that

$$\|g_{n_k} - g_{n_j}\|_1 \le C\|g_{n_k} - g_{n_j}\|_{L^2(\Omega)}.$$

Thus g belongs to $W_0^1(\Omega)$. It remains to show $g \in S$:

$$\begin{aligned}\|g\|_1 \le \|g - g_{n_k}\|_1 + \|g_{n_k}\|_1 &\overset{(11.13)}{\le} \varepsilon + C\|g_{n_k}\|_{L^2(\Omega)} \\ &\le \varepsilon + C\|g - g_{n_k}\|_{L^2(\Omega)} + C\|g\|_{L^2(\Omega)} \\ &\le \varepsilon(C+1) + C\|g\|_{L^2(\Omega)}.\end{aligned}$$

□

Corollary 11.14. *If $\Omega \subset \mathbb{R}^N$ is a smoothly bounded domain, then* $\dim \mathcal{H}^1(\Omega) < \infty$.

Proof. Integrating by parts gives, for $u \in \mathrm{Dom}\,(\Delta_1)$,

$$\begin{aligned}(\Delta_1 u, u) + \|u\|_{L^2}^2 &= (du, du) + (d^*u, d^*u) + \|u\|_{L^2}^2 \\ &= \|du\|_{L^2}^2 + \|d^*u\|_{L^2}^2 + \|u\|_{L^2}^2 \\ &\gtrsim \|u\|_1^2.\end{aligned}$$

If $\Delta_1 u = 0$, it follows that $\|u\|_{L^2}^2 \gtrsim \|u\|_1^2$. The previous Lemma finishes the proof. □