**Definition:** The Bergman projection \( B_{\Omega} = B \) is the Hilbert space orthogonal projection from \( L^2(\Omega) \to A^2(\Omega) \). It is achieved by integrating against the Bergman kernel, i.e., for \( f \in L^2(\Omega) \),

\[
B(f)(z) = \int_{\Omega} B(z, w) f(w) dV(w).
\]

(1) **Biholomorphic Transformation Law:** This exercise shows that a biholomorphism \( F : \Omega \to \tilde{\Omega} \) induces an isomorphism of the corresponding Bergman spaces, and allows us to deduce a transformation law for the Bergman kernel.

Recall \( J_C F \) denotes the \( n \times n \) matrix whose entries are the holomorphic partial derivatives of the components of \( F \). The determinant \( \det J_C F \) is called the **complex Jacobian**. Also, \( J_R F \) denotes the real \( 2n \times 2n \) matrix of partial derivatives which arises by thinking of \( F \) as a map from \( \mathbb{R}^{2n} \to \mathbb{R}^{2n} \). The determinant \( \det J_R F \) is called the **real Jacobian**.

(a) Prove that \( \det J_R F = |\det J_C F|^2 \).

(b) Let \( F : \Omega \to \tilde{\Omega} \) be a biholomorphism and \( \{\tilde{\phi}_\alpha\}_{\alpha \in A} \) be an orthonormal basis for \( A^2(\tilde{\Omega}) \).

Prove \( F \) induces an orthonormal basis \( \{\phi_\alpha\}_{\alpha \in A} \) for \( A^2(\Omega) \), where

\[
\phi_\alpha := \det J_C F \cdot \tilde{\phi}_\alpha \circ F. \tag{6.1}
\]

(c) Prove that

\[
B_{\Omega}(z, w) = \det J_C F(z) \cdot B_{\tilde{\Omega}}(F(z), F(w)) \cdot \det J_C F(w). \tag{6.2}
\]

(2) **Ramadanov’s theorem:** Let \( \{\Omega_j\}_{j \in \mathbb{Z}^+} \) be a set of domains which monotonically exhaust a bounded domain \( \Omega \subseteq \mathbb{C}^n \) from inside, and let \( B_j \) and \( B_\Omega \) denote the Bergman kernels of \( \Omega_j \) and \( \Omega \), respectively. Show that \( B_j(z, w) \to B_\Omega(z, w) \) uniformly on compact subsets of \( \Omega \times \Omega \).

(3) ** Explicit Bergman computations:** Let \( D \) denote the unit disc in \( \mathbb{C} \).

(a) Find an orthonormal basis for \( A^2(D) \).

(b) Show that the Bergman kernel on \( D \) is given by

\[
B(z, w) = \frac{1}{\pi(1 - z\bar{w})^2}.
\]

(c) Let \( f \) be an antiholomorphic \( L^2(D) \) function, i.e., \( \overline{f} \in A^2(D) \). Show that its Bergman projection \( B(f) = f(0) \).

(d) Show that the Bergman Kernel of the unit ball in \( \mathbb{C}^n \) is given by

\[
B(z, w) = \frac{n!}{\pi^n(1 - z\bar{w})^{n+1}}.
\]

(4) **Diagonal boundary behavior:** Let \( \delta_\Omega(z) = \delta(z) \) denote the distance to the boundary function of the domain

(a) Let \( \Omega \subset \mathbb{C}^n \) be a smoothly bounded domain. Prove that \( B_\Omega(z, z) \lesssim \delta(z)^{-n-1} \).

(b) Let \( \Omega \subset \mathbb{C}^n \) be any domain, with no assumption on boundary regularity. Prove that \( B_\Omega(z, z) \lesssim \delta(z)^{-2n} \).

(c) Give an example of a domain \( \Omega \subset \mathbb{C}^n \) with a non-smooth boundary point \( p \), and a path of points \( z \to p \) such that \( B_\Omega(z, z) \approx \delta(z)^{-2n} \) for all points along this path.
(5) \( L^p \) mapping range: By virtue of the fact that the \( B \) is defined as an orthogonal projection on a Hilbert space, it is an \( L^2 \) bounded operator. It is a very important question to understand how \( B \) acts on other \( L^p \) spaces. In the case of the unit disc \( \mathbb{D} \subset \mathbb{C} \), we will show that \( B \) is a bounded operator from \( L^p(\mathbb{D}) \to A^p(\mathbb{D}) \) for all \( 1 < p < \infty \).

(a) Young’s Test: This is a standard tool to show the \( L^p \) boundedness of an integral operator.

Let \( \Omega \subseteq \mathbb{C}^n \) be a domain, \( K \) be an a.e. positive, measurable function on \( \Omega \times \Omega \), and \( \mathcal{K} \) be the integral operator with kernel \( K \). Suppose that there exists a fixed constant \( C \), such that

\[
\int_{\Omega} K(z, w) dV(w) \leq C, \quad \forall z \in \Omega
\]

\[
\int_{\Omega} K(z, w) dV(z) \leq C, \quad \forall w \in \Omega.
\]

Then \( \mathcal{K} \) is a bounded operator on \( L^p(\Omega) \), for all \( p \in (1, \infty) \), and in fact,

\[
\|\mathcal{K}(f)\|_p \leq C \|f\|_p.
\]

(b) Schur’s Lemma: This is a souped-up version of Young’s Test, which lets us deal with more difficult kernels.

Let \( \Omega \subseteq \mathbb{C}^n \) be a domain, \( K \) be an a.e. positive, measurable function on \( \Omega \times \Omega \), and \( \mathcal{K} \) be the integral operator with kernel \( K \). Suppose there exists a positive auxiliary function \( h \) on \( \Omega \), and a number \( a > 0 \) such that for all \( \epsilon \in [0, a) \), the following estimates hold:

\[
\mathcal{K}(h^{-\epsilon})(z) := \int_{\Omega} K(z, w) h(w)^{-\epsilon} dV(w) \lesssim h(z)^{-\epsilon}
\]

\[
\mathcal{K}(h^{-\epsilon})(w) := \int_{\Omega} K(z, w) h(z)^{-\epsilon} dV(z) \lesssim h(w)^{-\epsilon}.
\]

Then \( \mathcal{K} \) is a bounded operator on \( L^p(\Omega) \), for all \( p \in (1, \infty) \).

(c) Show that the hypotheses of Schur’s Lemma are satisfied when the domain \( \Omega = \mathbb{D} \), the kernel \( K(z, w) = |B_\Omega(z, w)| \), and the auxiliary function \( h(z) = 1 - |z|^2 \).

Note: In part (c), \( h(z) \) is (essentially) the distance to the boundary function. In practice, this is usually a good choice for an auxiliary function. Using similar ideas coupled with more sophisticated machinery, it can be shown that the Bergman projection is an \( L^p \) bounded operator for \( 1 < p < \infty \) on a large class of smoothly bounded, pseudoconvex domains. For example, this is known to hold for all smooth bounded, strongly pseudoconvex domains. However, we will see later that such a result fails in general.