MSRI 2018: THE $\overline{\partial}$-PROBLEM IN THE TWENTY-FIRST CENTURY

June 12, 2018

Day-2 Exercises

(1) Suppose f and f^2 are real-valued harmonic functions on a domain $\Omega \subset \mathbb{R}^n$. Then f is constant.

(2) Let f be real-valued, harmonic and non-vanishing on $\Omega \subset \mathbb{R}^n$. Show that $|f|^p$ is subharmonic for all $p \geq 1$. In fact,
$$\Delta(|f|^p) = p(p-1)|f|^{p-2}|\nabla f|^2.$$

(3) Let f be holomorphic and non-vanishing on $\Omega \subset \mathbb{C}^n$. Show $\log|f|$ is harmonic.

(4) Let f be holomorphic and non-vanishing on $\Omega \subset \mathbb{C}^n$. Prove that $|f|^p$ is subharmonic for all $p > 0$. In fact,
$$\Delta(|f|^p) = p^2|f|^{p-2}|\partial f|^2.$$

(5) Let f be subharmonic on Ω. Show that for each $p \in \Omega$,
$$f(p) = \limsup_{x \to p} f(x).$$

(6) Let f be a subharmonic function on Ω which is not identically $-\infty$. Show that $f \in L^1_{\text{loc}}(\Omega)$.

(7) Give an example demonstrating the strict containment $PSH(\Omega) \subsetneq SH(\Omega)$.

(8) Show that the following series converges on the unit disc, and the function it represents has $\{|z| = 1\}$ as its natural boundary.
   (a) $\sum_{n=0}^{\infty} z^{n!}$
   (b) $\sum_{n=0}^{\infty} z^{2^n}$

(9) Let $\Omega_1 \subset \Omega_2 \subset \mathbb{C}^n$ be domains such that each $f \in \mathcal{O}(\Omega_1)$ extends to $\widetilde{f} \in \mathcal{O}(\Omega_2)$. Then show that
$$\{p \in \mathbb{C} : p = f(z), z \in \Omega_1\} = \{p \in \mathbb{C} : p = \widetilde{f}(z), z \in \Omega_2\}$$

(10) Let $f \in \mathcal{O}(\mathbb{D} \times \mathbb{D})$. Suppose $f(z_1, z_2) \neq 0$ if *either* $|z_1| < \frac{1}{2}$ or $|z_2| < \frac{1}{2}$. Show that f is non-zero everywhere on the unit ball.

(11) Find the domain of convergence of the following and the sum if possible
   (a) $\sum_{\nu_1,\nu_2=0}^{\infty} z_1^{\nu_1} z_2^{\nu_2}$.
   (b) $\sum_{k=0}^{\infty} (z_1 z_2)^k$.
   (c) $\sum_{\nu_1,\nu_2=0}^{\infty} \frac{\nu_1}{\nu_2!} z_1^{\nu_1} z_2^{\nu_2}$.
   (d) $\sum_{k=1}^{\infty} z^k w^{k!}$.
   (e) $\sum_{k=0}^{\infty} (z_1^2 + e^{ik} z_2^2)^k$
   (f) $\sum_{j=0}^{\infty} \left((2z_1)^j + \sum_{k=1}^{\infty} z_1^k\right) z_2^j$.

Definition *If X is a topological space then a function $u : X \to [-\infty, \infty)$ is called* ***upper semi-continuous*** *if for all $c \in \mathbb{R}$, the set $\{x \in X : u(x) < c\}$ is open in X. We will say that $u \in USC(X)$.*

(12) If $\Omega \subset \mathbb{R}^n$ is a domain, show the following
   (a) If $u \in USC(\Omega)$ and $c > 0$, then $cu \in USC(\Omega)$.
   (b) If $u_1, u_2 \in USC(\Omega)$, then $u_1 + u_2 \in USC(\Omega)$ and $u_1 u_2 \in USC(\Omega)$.

(c) If $u_1, \ldots, u_n$ are functions in $USC(\Omega)$, then the functions $v_1 = \max\{u_1, \ldots, u_n\}$ and $v_2 = \min\{u_1, \ldots, u_n\}$ are also in $USC(\Omega)$.
(d) If $\{u_j\}_{j=1}^{\infty}$ is a collection of functions in Ω then the function $v = \inf u_j$ is also in $USC(\Omega)$.
(e) Let $K \subset \Omega$ be compact and $u \in USC(\Omega)$. Show that there is an $M \in \mathbb{R}$ such that $u(x) < M$ for all $x \in K$.

(13) Let $\Omega \subset \mathbb{C}$ be a domain.
(a) If $u \in SH(\Omega)$ and $c > 0$ then $cu \in SH(\Omega)$.
(b) If $u_1, u_2 \in SH(\Omega)$ show that $u_1 + u_2 \in SH(\Omega)$.
(c) If $u_1, \ldots, u_n \in SH(\Omega)$ then $v = \max\{u_1, \ldots, u_n\} \in SH(\Omega)$.
(d) If $\{u_\alpha\} \subset SH(\Omega)$ and $v = \sup_\alpha u_\alpha$ is upper semi-continuous then $v \in SH(\Omega)$.
(e) If $\{u_j\}_{j=1}^{\infty} \subset SH(\Omega)$ and if $u_j \searrow u$, then $u \in SH(\Omega)$.