MSRI 2018: THE $\overline{\partial}$-PROBLEM IN THE TWENTY-FIRST CENTURY

June 14, 2018

Day-4 Exercises

We will use the following notation

$$D = \frac{\partial}{\partial z} \text{ and } \overline{D} = \frac{\partial}{\partial \bar{z}}.$$

Theorem 4.1 (Hörmander)**.** *Let $\Omega \subset \mathbb{C}$ be a domain and let $\varphi \in C^2(\Omega)$ be a strictly subharmonic function, that is, φ is real-valued and $\Delta\varphi > 0$. If g is a function on Ω such that*

$$\int_\Omega \frac{|g|^2}{\Delta\varphi} e^{-\varphi}\, dV < \infty,$$

then there is a function u on Ω such that $\overline{D}u = g$ and

$$\int_\Omega |u|^2 e^{-\varphi}\, dV \leq \int_\Omega \frac{|g|^2}{\Delta\varphi} e^{-\varphi}\, dV.$$

Prove this theorem using the following steps:

(1) We define the formal adjoint of $\frac{\partial}{\partial x}$ in $L^2(\Omega)$ as follows: For every $u \in L^2(\Omega)$ and for every $v \in C_0^2(\Omega)$, $\langle \frac{\partial}{\partial x} u, v\rangle = \langle u, \left(\frac{\partial}{\partial x}\right)^* v\rangle$. Calculate $\left(\frac{\partial}{\partial x}\right)^*$.

(2) Calculate the formal adjoint of $\overline{D}$ in $L^2(\Omega)$.

(3) Recall if φ is a continuous function then we define the weighted L^2 space with the weight $e^{-\varphi}$ as

$$L^2(\Omega, e^{-\varphi}) = \left\{ f : \Omega \to \mathbb{C} \ : \int_\Omega |f|^2 e^{-\varphi} dV < \infty \right\},$$

with the natural inner product on it

$$\langle f, g\rangle_\varphi = \int_\Omega f\overline{g} e^{-\varphi} dV.$$

Calculate the formal adjoint of $\overline{D}$ in $L^2(\Omega, e^{-\varphi})$, which is defined as follows: For every $u \in L^2(\Omega, e^{-\varphi})$ and $v \in C_0^2(\Omega)$,

$$\langle \overline{D}u, v\rangle_\varphi = \langle u, \overline{D}^* v\rangle_\varphi.$$

(4) *(A priori Estimates)* Let $\varphi \in C^2(\Omega)$ and let $\alpha \in C_0^2(\Omega)$. Show that

$$\langle (\Delta\varphi)\alpha, \alpha\rangle_\varphi \leq \|\overline{D}_\varphi^* \alpha\|_\varphi^2.$$

(Hint: Calculate the commutator $[\overline{D}, \overline{D}_\varphi^*]$)

(5) We have the operator $\overline{D}_\varphi^* : C_0^2(\Omega) \to L^2(\Omega, e^{-\varphi})$ whose range is a subspace of $L^2(\Omega, e^{-\varphi})$. Let $\mathscr{R} = \text{Range}(\overline{D}_\varphi^*) \subset C_0^1(\Omega) \subset L^2(\Omega, e^{-\varphi})$. So there is a linear functional $L : \mathscr{R} \to \mathbb{C}$ given by

$$L(\overline{D}_\varphi^* \alpha) = \langle \alpha, g\rangle_\varphi .$$

We now also assume that φ is a subharmonic function on Ω. Show that L is a bounded linear functional. (Hint: Use a priori estimates)

(6) Using Hahn-Banach theorem, this functional extends to whole of $L^2(\Omega, e^{-\varphi})$. Now use Riesz-Representation to prove Theorem 4.1.

Definition 4.2 (Hartogs' Triangle)**.** *The Hartogs' triangle* $\mathbb{H}$ *is defined as*

$$\mathbb{H} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$

Prove the following:

(1) $\mathbb{H}$ is biholomorphically equivalent to $\mathbb{D} \times \mathbb{D}^*$, where $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$.

(2) Show that the product of two pseudoconvex domains is pseudoconvex. Conclude that $\mathbb{H}$ is pseudoconvex.

(3) Every holomorphic function on a neighborhood of $\overline{\mathbb{H}}$ extends holomorphically to the bidisc.

(4) In fact, prove that each $f \in \mathcal{O}(\mathbb{H}) \cap C^\infty(\overline{\mathbb{H}})$ analytically continues to the bidisc.