We will use the following notation $D = \frac{\partial}{\partial z}$ and $\overline{D} = \frac{\partial}{\partial \overline{z}}$.

**Theorem 4.1** (Hörmander). Let $\Omega \subset \mathbb{C}$ be a domain and let $\varphi \in C^2(\Omega)$ be a strictly subharmonic function, that is, $\varphi$ is real-valued and $\Delta \varphi > 0$. If $g$ is a function on $\Omega$ such that

$$\int_{\Omega} \frac{|g|^2}{\Delta \varphi} e^{-\varphi} \, dV < \infty,$$

then there is a function $u$ on $\Omega$ such that $Du = g$ and

$$\int_{\Omega} |u|^2 e^{-\varphi} \, dV \leq \int_{\Omega} \frac{|g|^2}{\Delta \varphi} e^{-\varphi} \, dV.$$

Prove this theorem using the following steps:

1. We define the formal adjoint of $\frac{\partial}{\partial x}$ in $L^2(\Omega)$ as follows: For every $u \in L^2(\Omega)$ and for every $v \in C_0^2(\Omega)$, $\langle \frac{\partial}{\partial x} u, v \rangle = \langle u, (\frac{\partial}{\partial x})^* v \rangle$. Calculate $(\frac{\partial}{\partial x})^*$.

2. Calculate the formal adjoint of $D$ in $L^2(\Omega)$.

3. Recall if $\varphi$ is a continuous function then we define the weighted $L^2$ space with the weight $e^{-\varphi}$ as

$$L^2(\Omega, e^{-\varphi}) = \left\{ f : \Omega \to \mathbb{C} : \int_{\Omega} |f|^2 e^{-\varphi} \, dV < \infty \right\},$$

with the natural inner product on it

$$\langle f, g \rangle_{\varphi} = \int_{\Omega} f \overline{g} e^{-\varphi} \, dV.$$

Calculate the formal adjoint of $\overline{D}$ in $L^2(\Omega, e^{-\varphi})$, which is defined as follows: For every $u \in L^2(\Omega, e^{-\varphi})$ and $v \in C_0^2(\Omega)$,

$$\langle \overline{D} u, v \rangle_{\varphi} = \langle u, \overline{D}^* v \rangle_{\varphi}.$$

4. *(A priori Estimates)* Let $\varphi \in C^2(\Omega)$ and let $\alpha \in C_0^2(\Omega)$. Show that

$$\langle (\Delta \varphi) \alpha, \alpha \rangle_{\varphi} \leq ||\overline{D}_x^\varphi \alpha||_2^2.$$

(Hint: Calculate the commutator $[\overline{D}, \overline{D}_x^\varphi]$)

5. We have the operator $\overline{D}_{\varphi}^\varphi : C_0^2(\Omega) \to L^2(\Omega, e^{-\varphi})$ whose range is a subspace of $L^2(\Omega, e^{-\varphi})$.

Let $\mathcal{R} = \text{Range}(\overline{D}_{\varphi}^\varphi) \subset C_0^1(\Omega) \subset L^2(\Omega, e^{-\varphi})$. So there is a linear functional $L : \mathcal{R} \to \mathbb{C}$ given by

$$L(\overline{D}_{\varphi}^\varphi \alpha) = \langle \alpha, g \rangle_{\varphi}.$$

We now also assume that $\varphi$ is a subharmonic function on $\Omega$. Show that $L$ is a bounded linear functional. (Hint: Use a priori estimates)

**Definition 4.2** (Hartogs’ Triangle). The Hartogs’ triangle $H$ is defined as

$$H = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < |z_2| < 1\}.$$ 

Prove the following:

1. $H$ is biholomorphically equivalent to $D \times D^*$, where $D^* = D \setminus \{0\}$.
2. Show that the product of two pseudoconvex domains is pseudoconvex. Conclude that $H$ is pseudoconvex.
3. Every holomorphic function on a neighborhood of $H$ extends holomorphically to the bidisc.
4. In fact, prove that each $f \in \mathcal{O}(H) \cap C^\infty(H)$ analytically continues to the bidisc.