Exercises on Stable Hypersurfaces of Constant Mean Curvature

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0.1 Variation formulas

Let $\Sigma^n \subset (M^{n+1}, g)$ be a smooth compact hypersurface without boundary. Consider the variation of Σ along its normal direction $F(\cdot, t) : \Sigma^n \to M^{n+1}$, for $-\epsilon < t < \epsilon$, satisfying

$$\frac{\partial}{\partial t} F(x,t) = \eta(x,t)\nu(x,t)$$
$$F(\Sigma, 0) = \Sigma,$$

where $\nu(x,t)$ is the outward-pointing unit normal to $\Sigma_t := \{F(x,t) : x \in \Sigma\}$.

Exercise 1. *Denote by $d\mu(x,t)$ and $H(x,t)$ the volume measure and the mean curvature of Σ_t respectively; note that the mean curvature vector will be $\mathbf{H} = -H\nu$. If the first fundamental form (induced metric) on Σ_t is $h(x,t)$, then the second fundamental form on Σ_t is given by $A_{\Sigma_t}(X,Y) = g(D_X Y, \nu)$, so that $H = -tr_h A_{\Sigma_t}$. Let $A(t) = \int_\Sigma d\mu(x,t)$ be the n-volume of Σ_t. Then the first variation formula of $A(t)$ is*

$$A'(t) = \int_\Sigma H(x,t)\eta(x,t)\, d\mu(x,t).$$

If the mean curvature of Σ is zero, or if Σ has nonzero constant mean curvature and $\int_\Sigma \eta(x,t)\, d\mu(x,t) = 0$ for all t, then the second variation formula is

$$A''(0) = \int_\Sigma \eta(x,0) L_\Sigma \eta(x,0)\, d\mu(x,0),$$

where $L_{\Sigma_t}\eta(x,t) = -\Delta_{\Sigma_t}\eta(x,t) - (\|A_{\Sigma_t}\|_g^2 + Ric_g(\nu(x,t), \nu(x,t)))\eta(x,t)$. Alternatively, the pointwise variation formulas are

$$\frac{d}{dt} d\mu(x,t) = H(x,t)\eta(x,t)\, d\mu(x,t),$$
$$\frac{d}{dt} H(x,t) = L_{\Sigma_t}\eta(x,t).$$

Definition. *(1) If Σ is a minimal hypersurface, i.e., $H = 0$, then Σ is called stable if $A''(0) \geq 0$ for any variation $\eta \in C^2(\Sigma)$.*

(2) If Σ is a hypersurface with nonzero constant mean curvature, then Σ is called stable if $A''(0) \geq 0$ for any variation $\eta \in C^2(\Sigma)$ with $\int_\Sigma \eta \, d\mu = 0$.

Remark. A stable minimal n-dimensional hypersurface minimizes the n-volume among nearby hypersurfaces. For a hypersurface with nonzero constant mean curvature, the stability implies that it minimizes the n-volume among nearby hypersurfaces which enclose the same $(n+1)$-volume in M.

0.2 Stable round spheres in Schwarzschild solutions

Let $g_m = \left(1 + \frac{m}{2|x|}\right)^4 \delta = \phi^4 \delta$ be a three-dimensional spatial Schwarzschild metric of mass $m > 0$ on $\mathbb{R}^3 \backslash \{0\}$. Note that g_m is conformally flat and complete. Let $S_r = \{x \in \mathbb{R}^3 : |x| = r\}$ denote the round spheres centered at the origin, and let Δ_{S_r} be the Laplacian for the induced metric on S_r.

Exercise 2. *The round spheres S_r are umbilic and have constant mean curvature $\left(2 - \frac{m}{r}\right) \frac{\phi^{-3}}{r}$. ($S_{\frac{m}{2}}$ is the minimal surface, i.e., the horizon of the Schwarzschild solution.)*

The stability operator $L_{S_r} = -\Delta_{S_r} - (\|A_{S_r}\|^2_{g_m} + Ric_{g_m}(\nu, \nu))$ is $-\phi^{-4} r^{-2} \Delta_{\mathbb{S}^2} + \frac{-4r^2 + 8rm - m^2}{2r^4 \phi^6}$ where $\Delta_{\mathbb{S}^2}$ is the Laplacian for the standard unit sphere. Hence L_{S_r} and $\Delta_{\mathbb{S}^2}$ have the same eigenfunctions.

The lowest eigenvalue of L_{S_r} is $\lambda_0 = \frac{-4r^2 + 8rm - m^2}{2r^4 \phi^6}$ with the eigenspace space spanned by constants, and the next eigenvalues are $\lambda_1 = \lambda_2 = \lambda_3 = \frac{6m}{r^3 \phi^6}$ with the eigenspace spanned by $\{x^1, x^2, x^3\}$. Therefore, $\{S_r\}$ form a foliation of stable surfaces of constant mean curvature in the Schwarzschild solution.

To compute the above exercise, you may find it convenient to use the transformation formulas for conformal metrics. If two metrics on M^n are related by $\bar{g} = u^{\frac{4}{n-2}} g$, then

$$\overline{H} = u^{\frac{-2}{n-2}} \left(H + \tfrac{2(n-1)}{n-2} u^{-1} \nabla_\nu u\right),$$

$$\overline{R}_{ij} = R_{ij} - 2(\log u)_{ij} + \frac{4}{n-2} (\log u)_i (\log u)_j - \left(\frac{2}{n-2} \Delta(\log u) + \frac{4}{n-2} |\nabla \log u|^2\right) g_{ij}.$$