0.1 Variation formulas

Let $\Sigma^n \subset (M^{n+1}, g)$ be a smooth compact hypersurface without boundary. Consider the variation of $\Sigma$ along its normal direction $F(\cdot, t) : \Sigma^n \to M^{n+1}$, for $-\epsilon < t < \epsilon$, satisfying

$$\frac{\partial}{\partial t} F(x, t) = \eta(x, t) \nu(x, t)$$

$$F(\Sigma, 0) = \Sigma,$$

where $\nu(x, t)$ is the outward-pointing unit normal to $\Sigma_t := \{F(x, t) : x \in \Sigma\}$.

**Exercise 1.** Denote by $d\mu(x, t)$ and $H(x, t)$ the volume measure and the mean curvature of $\Sigma_t$ respectively; note that the mean curvature vector will be $H = -H\nu$. If the first fundamental form (induced metric) on $\Sigma_t$ is $h(x, t)$, then the second fundamental form on $\Sigma_t$ is given by $A_{\Sigma_t}(X, Y) = g(D_X Y, \nu)$, so that $H = -tr_h A_{\Sigma_t}$. Let $A(t) = \int_\Sigma d\mu(x, t)$ be the $n$-volume of $\Sigma_t$. Then the first variation formula of $A(t)$ is

$$A'(t) = \int_\Sigma H(x, t)\eta(x, t) d\mu(x, t).$$

If the mean curvature of $\Sigma$ is zero, or if $\Sigma$ has nonzero constant mean curvature and $\int_\Sigma \eta(x, t) d\mu(x, t) = 0$ for all $t$, then the second variation formula is

$$A''(0) = \int_\Sigma \eta(x, 0)L_{\Sigma_t}\eta(x, 0) d\mu(x, 0),$$

where $L_{\Sigma_t}\eta(x, t) = -\Delta_{\Sigma_t}\eta(x, t) - (\|A_{\Sigma_t}\|^2 g + \text{Ric}_g(\nu(x, t), \nu(x, t)))\eta(x, t)$. Alternatively, the pointwise variation formulas are

$$\frac{d}{dt}d\mu(x, t) = H(x, t)\eta(x, t) d\mu(x, t),$$

$$\frac{d}{dt}H(x, t) = L_{\Sigma_t}\eta(x, t).$$
Definition.  (1) If \( \Sigma \) is a minimal hypersurface, i.e., \( H = 0 \), then \( \Sigma \) is called stable if \( A''(0) \geq 0 \) for any variation \( \eta \in C^2(\Sigma) \).

(2) If \( \Sigma \) is a hypersurface with nonzero constant mean curvature, then \( \Sigma \) is called stable if \( A''(0) \geq 0 \) for any variation \( \eta \in C^2(\Sigma) \) with \( \int_{\Sigma} \eta d\mu = 0 \).

Remark. A stable minimal \( n \)-dimensional hypersurface minimizes the \( n \)-volume among nearby hypersurfaces. For a hypersurface with nonzero constant mean curvature, the stability implies that it minimizes the \( n \)-volume among nearby hypersurfaces which enclose the same \((n+1)\)-volume in \( M \).

0.2 Stable round spheres in Schwarzschild solutions

Let \( g_m = \left(1 + \frac{m^2}{2|x|^2}\right)^4 \delta = \phi^4 \delta \) be a three-dimensional spatial Schwarzschild metric of mass \( m > 0 \) on \( \mathbb{R}^3 \setminus \{0\} \). Note that \( g_m \) is conformally flat and complete. Let \( S_r = \{x \in \mathbb{R}^3 : |x| = r\} \) denote the round spheres centered at the origin, and let \( \Delta_{S_r} \) be the Laplacian for the induced metric on \( S_r \).

Exercise 2. The round spheres \( S_r \) are umbilic and have constant mean curvature \( (2 - \frac{m}{r}) \frac{\phi^{-3}}{r} \). (\( S_{\frac{m}{2}} \) is the minimal surface, i.e., the horizon of the Schwarzschild solution.)

The stability operator \( L_{S_r} = -\Delta_{S_r} - \|A_{S_r}\|_{g_m}^2 + \text{Ric}_{g_m}(\nu, \nu) \) is \(-\phi^{-4}r^{-2}\Delta_{S^2} + \frac{-4r^2 + 8rm - m^2}{2r^4 \phi^6} \) where \( \Delta_{S^2} \) is the Laplacian for the standard unit sphere. Hence \( L_{S_r} \) and \( \Delta_{S^2} \) have the same eigenfunctions.

The lowest eigenvalue of \( L_{S_r} \) is \( \lambda_0 = \frac{-4r^2 + 8rm - m^2}{2r^4 \phi^6} \) with the eigenspace spanned by constants, and the next eigenvalues are \( \lambda_1 = \lambda_2 = \lambda_3 = \frac{6m}{r^3 \phi^6} \) with the eigenspace spanned by \( \{x^1, x^2, x^3\} \). Therefore, \( \{S_r\} \) form a foliation of stable surfaces of constant mean curvature in the Schwarzschild solution.

To compute the above exercise, you may find it convenient to use the transformation formulas for conformal metrics. If two metrics on \( M^n \) are related by \( \tilde{g} = u^{\frac{4}{n-2}} g \), then

\[
\tilde{H} = u^{\frac{4}{n-2}} (H + \frac{2(n-1)}{n-2} u^{-1} \nabla \nu u),
\]

\[
\tilde{R}_{ij} = R_{ij} - 2(\log u)_{ij} + \frac{4}{n-2} (\log u)_i (\log u)_j - \left( \frac{2}{n-2} \Delta (\log u) + \frac{4}{n-2} \nabla \log u \right)^2 g_{ij}.
\]