1. Lecture 5: Differential forms and de Rham cohomology

Exercise 1.1. *Verify that the only local information a holomorphic form carries is its order of vanishing. That is, show that if $\omega \in \Omega^{(1,0)}(X)$ is holomorphic, then for each $p \in X$ there is a coordinate chart $(U, \tilde{U}, \phi)$ with $p \in U$ and $\phi(p) = 0$, so that $(\phi^{-1})^*\omega = z^n dz$ for some integer $n \geq 0$.*

Exercise 1.2. *Show that $\mathbb{S}^2$ admits no non-vanishing holomorphic forms.*

Exercise 1.3. *Show that if X is a Riemann surface for which $H^1(X) = 0$, then every harmonic function $u : X \to \mathbb{R}$ has a (unique up to addition of a constant) harmonic conjugate $v : X \to \mathbb{R}$ so that $f = u + iv : X \to \mathbb{C}$ is holomorphic.*

Exercise 1.4. *Check that if g is a metric on X whose conformal structure $[g]$ agrees with the Riemann surface structure, then*

$$\Delta f = (\Delta_g f) dVol_g$$

where $dVol_g$ is the Volume form of g and Δ_g is the Laplace-Beltrami operator of the metric.

Exercise 1.5. *A* complex projective structure *is a second order differential operator that makes sense on certain complex line bundles over a Riemann surface.*

(1) *Show that if $\phi : U \to U'$ is holomorphic for simply-connected $U, U' \subset \mathbb{C}$, then for a function $f \in C^\infty(U', \mathbb{C})$*

$$Z \cdot \left(Z \cdot \left(\frac{f \circ \phi}{(\phi')^{1/2}} \right) \right) = ((Z \cdot (Z \cdot f)) \circ \phi)(\phi')^{3/2} - \frac{1}{2} S(\phi) \frac{f \circ \phi}{(\phi')^{1/2}}$$

where

$$S(\phi) = \left(\frac{\phi''}{\phi'} \right)' - \frac{1}{2} \left(\frac{\phi''}{\phi'} \right)^2$$

is the Schwarzian derivative of ϕ (here $'$ means to apply Z).

(2) *Consider linearly independent holomorphic solutions to the complex ODE*

$$Z^2 - \frac{1}{2} S(\phi)$$

how do the relate to the solutions 1 and z of the ODE Z^2?

(3) *Show that $S(\phi) = 0$ if and only if ϕ is a fractional linear transformation – i.e $\phi(z) = \frac{az+b}{cz+d}$.*

(4) *Show that $S(\phi \circ \psi) = (S(\phi) \circ \psi)(\psi')^2 + S(\psi)$.*

(5) *Argue that on a Riemann surface X there are complex line bundles $L_{-1/2}$ and $L_{3/2}$ over X so that one can define a second order differential operator*

$$\mathcal{P} : L_{-1/2} \to L_{3/2}$$

so that in local coordinates $\mathcal{P}$ is of the form $\mathcal{P}(f(z)dz^{-1/2}) = Z(\cdot Z \cdot f)dz^{3/2}$. Such an operator is called a complex projective structure. (Hint: Think about how holomorphic solutions to second order holomorphic ODEs transform under holomorphic change of coordinates).

(6) *Show that the space of complex projective structures on X is an affine space modeled on the space of holomorphic quadratic differentials (i.e. holomophic sections of the bundle L_2 of quadratic differentials).*

(7) *Show that the Riemann sphere $\mathbb{S}^2$ has a unique complex projective structure (modulo equivalence).*