1. Lecture 5: Differential forms and de Rham cohomology

Exercise 1.1. Verify that the only local information a holomorphic form carries is its order of vanishing. That is, show that if \( \omega \in \Omega^{(1,0)}(X) \) is holomorphic, then for each \( p \in X \) there is a coordinate chart \((U, \tilde{U}, \phi)\) with \( p \in U \) and \( \phi(p) = 0 \), so that \((\phi^{-1})^*\omega = z^n dz\) for some integer \( n \geq 0 \).

Exercise 1.2. Show that \( S^2 \) admits no non-vanishing holomorphic forms.

Exercise 1.3. Show that if \( X \) is a Riemann surface for which \( H^1(X) = 0 \), then every harmonic function \( u : X \to \mathbb{R} \) has a (unique up to addition of a constant) harmonic conjugate \( v : X \to \mathbb{R} \) so that \( f = u + iv : X \to \mathbb{C} \) is holomorphic.

Exercise 1.4. Check that if \( g \) is a metric on \( X \) whose conformal structure \([g]\) agrees with the Riemann surface structure, then
\[
\Delta f = (\Delta_g f) d\text{Vol}_g
\]
where \( d\text{Vol}_g \) is the Volume form of \( g \) and \( \Delta_g \) is the Laplace-Beltrami operator of the metric.

Exercise 1.5. A complex projective structure is a second order differential operator that makes sense on certain complex line bundles over a Riemann surface.

1. Show that if \( \phi : U \to U' \) is holomorphic for simply-connected \( U, U' \subset \mathbb{C} \), then for a function \( f \in C^\infty(U', \mathbb{C}) \)
\[
Z \cdot \left(Z \cdot \left(f \circ \phi \left( \frac{1}{(\phi')^{1/2}} \right) \right) \right) = ((Z \cdot (Z \cdot f)) \circ \phi)'(\phi')^{3/2} - \frac{1}{2} S(\phi) f \circ \phi
\]
where
\[
S(\phi) = \left( \frac{\phi''}{\phi'} \right)' - \frac{1}{2} \left( \frac{\phi''}{\phi'} \right)^2
\]
is the Schwarzian derivative of \( \phi \) (here \( ' \) means to apply \( Z \)).

2. Consider linearly independent holomorphic solutions to the complex ODE
\[
Z^2 - \frac{1}{2} S(\phi)
\]
how do these relate to the solutions 1 and 3 of the ODE \( Z^2 \)?

3. Show that \( S(\phi) = 0 \) if and only if \( \phi \) is a fractional linear transformation – i.e. \( \phi(z) = \frac{az+b}{cz+d} \).

4. Show that \( S(\phi \circ \psi) = (S(\phi) \circ \psi)(\psi')^2 + S(\psi) \).

5. Argue that on a Riemann surface \( X \) there are complex line bundles \( L_{-1/2} \) and \( L_{3/2} \) over \( X \) so that one can define a second order differential operator
\[
P : L_{-1/2} \to L_{3/2}
\]
so that in local coordinates \( P \) is of the form \( P(f(z)dz^{-1/2}) = Z(Z \cdot f)dz^{3/2} \). Such an operator is called a complex projective structure. (Hint: Think about how holomorphic solutions to second order holomorphic ODEs transform under holomorphic change of coordinates).

6. Show that the space of complex projective structures on \( X \) is an affine space modeled on the space of holomorphic quadratic differentials (i.e. holomorphic sections of the bundle \( L_2 \) of quadratic differentials).

7. Show that the Riemann sphere \( S^2 \) has a unique complex projective structure (modulo equivalence).