1. Lecture 7: Poisson Equation

Exercise 1.1. *Relate the Poincaré inequality to Dirichlet and Neumann eigenvalue problems. Here Ω is a fixed bounded smooth domain in $\mathbb{R}^2$.*

(1) *Suppose that ϕ is a smooth function on $\bar{\Omega}$ so that $\int_\Omega \phi = 0$ and*

$$R[\phi_N] = \frac{\int_\Omega |\nabla \phi_N|^2 dA}{\int_\Omega \phi_N^2 dA} \leq \frac{\int_\Omega |\nabla \psi|^2 dA}{\int_\Omega \psi^2 dA}$$

for all smooth ψ in Ω which satisfy $\int_\Omega \psi = 0$ – that is, $R[\phi]$ is the best possible constant in the Poincaré inequality for functions with orthogonal to the constants. Show that ϕ satisfies the following equation with Neumann boundary values

$$\Delta \phi_N = -R[\phi_N]\phi \text{ on } \Omega, \partial_\nu \phi_N = 0 \text{ on } \partial\Omega$$

where here $\partial_\nu \phi$ is the outward normal derivative of ϕ_N.

(2) *For Ω convex, show that such ϕ_N exists. Is it unique?*

(3) *Suppose that ϕ is a smooth function on $\bar{\Omega}$ with compact support and*

$$R[\phi_D] = \frac{\int_\Omega |\nabla \phi_D|^2 dA}{\int_\Omega \phi^2 dA} \leq \frac{\int_\Omega |\nabla \psi|^2 dA}{\int_\Omega \psi^2 dA}$$

for all smooth ψ in Ω of compact – that is, $R[\phi]$ is the best possible constant in the Poincaré inequality for functions with compact support. Show that ϕ_D satisfies the following equation with Dirichlet boundary values

$$\Delta \phi_D = -R[\phi_D]\phi_D \text{ on } \Omega, \phi_D = 0 \text{ on } \partial\Omega.$$

(4) *For Ω convex, show that ϕ_D exists and is unique up to multiplying by a constant.*

(5) *Analyze ϕ_D and ϕ_N when Ω is a disk.*

Exercise 1.2. *Consider the non-convex domains*

$$\Omega_\epsilon = \left\{|z-2|^2 < 1\right\} \cup \left\{|z+2|^2 < 1\right\} \cup \{|Im(z)| < \epsilon, |\Re(z)| < 2\} \subset \mathbb{C}.$$

Show that the Poincaré inequality for functions which are orthogonal to the constatns cannot hold with a uniform constant for all of these domains.

Exercise 1.3. *Let ρ be a smooth form with compact support on $\mathbb{C}$ so that $\int_\mathbb{C} \rho = C \neq 0$. Show that if $\Delta u = \rho$ for some smooth function u, then*

$$\frac{1}{r}\int_{\partial D_r} u = 2\pi C \log r.$$

Conclude that there is no solution that vanishes at ∞ to the Poisson equation $\Delta u = \rho$. What happens for $\mathbb{D}$?

Exercise 1.4. *For a bounded open subset $\Omega \subset \mathbb{R}^2$ and fixed smooth 2-form ρ on Ω with compact support, explictly construct a smooth function ϕ so that $\Delta\phi = \Omega$. Hint: Use the kernel $K(x,y) = \frac{1}{2\pi}\log|x-y|$.*