1. Lecture 7: Poisson Equation

Exercise 1.1. Relate the Poincaré inequality to Dirichlet and Neumann eigenvalue problems. Here $\Omega$ is a fixed bounded smooth domain in $\mathbb{R}^2$.

1. Suppose that $\phi$ is a smooth function on $\Omega$ so that $\int_\Omega \phi = 0$ and

$$ R[\phi_N] = \frac{\int_\Omega |\nabla \phi_N|^2 dA}{\int_\Omega \phi_N^2 dA} \leq \frac{\int_\Omega |\nabla \psi|^2 dA}{\int_\Omega \psi^2 dA} $$

for all smooth $\psi$ in $\Omega$ which satisfy $\int_\Omega \psi = 0$ – that is, $R[\phi]$ is the best possible constant in the Poincaré inequality for functions with orthogonal to the constants. Show that $\phi$ satisfies the following equation with Neumann boundary values

$$ \Delta \phi_N = -R[\phi_N]\phi_N \text{ on } \Omega, \partial_v \phi_N = 0 \text{ on } \partial \Omega $$

where here $\partial_v \phi$ is the outward normal derivative of $\phi_N$.

2. For $\Omega$ convex, show that such $\phi_N$ exists. Is it unique?

3. Suppose that $\phi$ is a smooth function on $\Omega$ with compact support and

$$ R[\phi_D] = \frac{\int_\Omega |\nabla \phi_D|^2 dA}{\int_\Omega \phi_D^2 dA} \leq \frac{\int_\Omega |\nabla \psi|^2 dA}{\int_\Omega \psi^2 dA} $$

for all smooth $\psi$ in $\Omega$ of compact – that is, $R[\phi]$ is the best possible constant in the Poincaré inequality for functions with compact support. Show that $\phi_D$ satisfies the following equation with Dirichlet boundary values

$$ \Delta \phi_D = -R[\phi_D]\phi_D \text{ on } \Omega, \phi_D = 0 \text{ on } \partial \Omega. $$

4. For $\Omega$ convex, show that $\phi_D$ exists and is unique up to multiplying by a constant.

5. Analyze $\phi_D$ and $\phi_N$ when $\Omega$ is a disk.

Exercise 1.2. Consider the non-convex domains

$$ \Omega = \{|z - 2|^2 < 1\} \cup \{|z + 2|^2 < 1\} \cup \{|Im(z)| < \epsilon, |\Re(z)| < 2\} \subset \mathbb{C}. $$

Show that the Poincaré inequality for functions which are orthogonal to the constants cannot hold with a uniform constant for all of these domains.

Exercise 1.3. Let $\rho$ be a smooth form with compact support on $\mathbb{C}$ so that $\int_\Omega \rho = C \neq 0$. Show that if $\Delta u = \rho$ for some smooth function $u$, then

$$ \frac{1}{r} \int_{\partial D_r} u = 2\pi C \log r. $$

Conclude that there is no solution that vanishes at $\infty$ to the Poisson equation $\Delta u = \rho$. What happens for $D$?

Exercise 1.4. For a bounded open subset $\Omega \subset \mathbb{R}^2$ and fixed smooth 2-form $\rho$ on $\Omega$ with compact support, explicitly construct a smooth function $\phi$ so that $\Delta \phi = \rho$. Hint: Use the kernel $K(x, y) = \frac{1}{2\pi} \log |x - y|$. 