Exercises for Lecture 1
(1) Let \(C^{n-2} \subset C^n\) be the linear subspace spanned by the first \(n-2\) coordinates. Show that any holomorphic function \(f : C \setminus C^{n-2} \to C\) admits a holomorphic extension \(\tilde{f} : C^n \to C\).

(2) Let \(f\) be a holomorphic function on \(C^n\). What can you say about the \((1,1)\)-form \(\partial \overline{\partial} \log |f|^2\)?

(3) Study the analytic sets \(S_1 = \{z^2(z + 1) - w^2 = 0\}\) and \(S_2 = \{z^3 - w^2 = 0\}\) in \(C^2\).

- Draw pictures of \(S_1 \cap \mathbb{R}^2\) and \(S_2 \cap \mathbb{R}^2\). How accurate are these?
- Write \(S_i\) \((i = 1, 2)\) as the image of a holomorphic map \(f_i : C \to C^2\), with \(f_2\) bijective.
- Show that the analytic set \(\{xw - yz = 0, xz - y^2 = 0, yw - z^2 = 0\}\) has complex codimension two in \(C^4\) but cannot be defined by fewer than three holomorphic equations near the origin.

[An easier but less satisfactory example: Show that the codimension one subset \(\{xy + zw = 0\}\) cannot be cut out by fewer than two holomorphic equations.]

(4) Study the Legendre family \(E_t : w^2 = z(z + 1)(z + t)\) of elliptic curves in \(C^2\).

- Use the presentation of \(O_{P^1}(-1)\) in terms of charts and transition functions given in class to verify directly that \(O_{P^1}(−1)\) has no nonzero global holomorphic sections.

(5) Show that the holomorphic map \(S(P^1 \times P^1)\) is bijective.

- Draw pictures of \(E_t \cap \mathbb{R}^2\). For which values of \(t\) is \(E_t\) not smooth?
- Can you visualize what happens to the Riemann surface \(E_t\) as \(t\) approaches these values?

Exercises for Lecture 2
(1) Let \((z, w)\) be affine coordinates on \(C^2\) and \((x : y)\) homogeneous coordinates on \(P^1\). Show that the equation \(zy = wx\) cuts out a smooth complex hypersurface \(H\) in \(C^2 \times P^1\). Describe the fibers of the holomorphic map \(H \to C^2\) given by projection onto the first factor.

(2) Show that the holomorphic map \(P^1 \times P^1 \to P^2\) given by \(((x_i, y_j)) \mapsto (x_i y_j)\) is an embedding. Show that the image is the zero locus of a homogeneous quadratic polynomial.

(3) This exercise looks at holomorphic maps between projective spaces and tori.

- Show that a holomorphic map from \(P^m\) to a torus \(C^m / \Lambda\) is constant.
- Show that there is a 1-to-1 correspondence between holomorphic maps from a Riemann surface to \(P^1\) and meromorphic functions on the Riemann surface.

- Given a lattice \(\Lambda \subset C\), show that the series
  \[
  \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus 0} \left( \frac{1}{(z + \lambda)^2} - \frac{1}{\lambda^2} \right)
  \]
  converges and defines a nonconstant meromorphic function on \(C / \Lambda\).

(4) Study the Legendre family \(E_t : w^2 = z(z + 1)(z + t)\) of elliptic curves in \(C^2\).

- Use the fact that \(E(t)\) is equal to \(\overline{E(\bar{t})}\) to deduce that the ratio \(\frac{1}{4} \text{Area}(U \cap B_r(0))\) is monotone increasing in \(r\). Here \(B_r(0)\) denotes the standard ball of radius \(r\) in \(C^2\).
- Show that \(U\) is a minimal surface in \(C^2\).

Exercises for Lecture 3
(1) Use the presentation of \(O_{P^n}(-1)\) in terms of charts and transition functions given in class to verify directly that \(O_{P^n}(−1)\) has no nonzero global holomorphic sections.

(2) The tangent bundle of any Riemann surface can be given the structure of a holomorphic line bundle in a natural way. As such, \(T\mathbb{P}^1 = O_{P^1}(k)\) for some \(k \in \mathbb{Z}\). Find \(k\).

(3) Let \(L\) be a holomorphic line bundle on some compact complex manifold.
- Show that if both \( L \) and \( L^* \) have a nonzero global holomorphic section, then both \( L \) and \( L^* \) are isomorphic to the trivial line bundle as holomorphic line bundles.
- Denoting the zero section of a line bundle by 0, prove that \( L \setminus 0 \) and \( L^* \setminus 0 \) are isomorphic as holomorphic fiber bundles with fiber \( \mathbb{C}^* \), but not as holomorphic principal \( \mathbb{C}^* \)-bundles.
- Show that \( L \) and \( L^* \) are always isomorphic as smooth real rank two vector bundles, but not as smooth complex rank one vector bundles.

(4) Courtesy of [http://en.wikipedia.org/wiki/Theta_function](http://en.wikipedia.org/wiki/Theta_function) Let \( \tau \) be a complex number in the upper half plane. Let \( \Lambda = \mathbb{Z} + \mathbb{Z} \tau \). Show that the Jacobi theta function \( J(z, \tau) \) can be viewed as a holomorphic section of a holomorphic line bundle \( L \) on \( \mathbb{C}/\Lambda \). (From Section 9 of the Wikipedia page, \( L \) is isomorphic to \( \mathcal{O}_{\mathbb{C}/\Lambda}(\frac{1+i\tau}{2}) \), so that \( J(z, \tau) \) is the unique section of \( L \).)

**Exercises for Lecture 4**

1. Let \( D \) be a compact complex curve in a compact complex surface \( M \). Let \( h \) be a Hermitian metric of positive curvature on \( \mathcal{O}_M(D) \). Then every compact complex curve in \( M \) intersects \( D \).

2. Let \( f \) and \( g \) be homogeneous polynomials of degree \( m \) and \( n \) on \( \mathbb{C}^3 \). Consider the associated complex algebraic curves \( F = \{ f = 0 \} \) and \( G = \{ g = 0 \} \) in \( \mathbb{P}^2 \). Use integration by parts to show that if \( #(F \cap G) < \infty \), then \( mn = \sum_{p \in F \cap G} \gamma_p \), where \( \gamma_p \) is a suitably defined “residue” at \( p \). Then compute \( \gamma_p \) in some examples, preferably ones where \( p \) is a bad singularity of both \( F \) and \( G \).

3. This is a continuation of Exercise 4 on the previous sheet. The total space of the line bundle \( L \) of which Jacobi’s theta function \( w = J(z, \tau) \) is a section can be written as \( (\mathbb{C} \times \mathbb{C})/\sim \), where we identify \( (z, w) \sim (z + a + b\tau, e^{-\pi ib^2\tau - 2\pi ibz}w) \) for all \( a, b \in \mathbb{Z} \). Use this fact to construct a Hermitian metric \( h \) on \( L \) such that \( \frac{1}{2\pi} \omega_h = \frac{1}{3} \frac{dz \wedge d\bar{z}}{|dz|^2} \), the preferred generator of \( H^2(\mathbb{C}/\Lambda, \mathbb{Z}) \).

4. This exercise is meant as an introduction to Grauert’s contraction theorem. We will construct some examples of rigid curves with outward convex neighborhoods on surfaces.

- Let \( S \) be the affine surface \( z^2 = xy \) in \( \mathbb{C}^3 \), which is singular precisely at the origin. Blow up the origin in \( \mathbb{C}^3 \) by passing to the submanifold \( H \subset \mathbb{C}^3 \times \mathbb{P}^2 \) cut out by \( xv = yu \) and \( yw = zv \) in affine coordinates \((x, y, z)\) on \( \mathbb{C}^3 \) and homogeneous coordinates \((u : v : w)\) on \( \mathbb{P}^2 \). Show that if \( p : H \to \mathbb{C}^3 \) denotes the projection, then \( p^{-1}(S) = p^{-1}(0) \cup \hat{S} \), where \( \hat{S} \) is a smooth surface containing a rigid curve isomorphic to \( \mathbb{P}^1 \): the intersection of \( \hat{S} \) with \( p^{-1}(0) = \mathbb{P}^2 \).

- Repeat with the surface \( z^3 = xy \). What changes?

**Exercises for Lecture 5**

1. Let \( L \) be a holomorphic line bundle over a compact complex manifold \( M \). Let \( V \) denote the space of global holomorphic sections of \( L \). Show that the following hold:

- If \( L \) is basepoint free, then \( \varphi_L^* \mathcal{O}(1) = L \).
- If \( L \) is basepoint free, then the divisors \( s^{-1}(0) \) for \( s \in V \) are precisely the preimages under \( \varphi_L \) of the intersections of \( \varphi_L(M) \) with linear hyperplanes in \( \mathbb{P}(V^*) \).
- If \( L \) is very ample, then \( L^k \) is very ample for every \( k \in \mathbb{N} \).

2. Let \( C \) be the image of \( \mathbb{C}/\Lambda \) under the embedding into \( \mathbb{P}^2 \) associated with \( \mathcal{O}(3p) \). Prove that \( p_1, p_2, p_3 \in C \) lie on a straight line if and only if \( \mathcal{O}(p_1) \otimes \mathcal{O}(p_2) \otimes \mathcal{O}(p_3) \cong \mathcal{O}(3p) \). One can show that this is equivalent to \( p_1 + p_2 + p_3 = 3p \) in \( \mathbb{C}/\Lambda \). Which points \( q \in C \) satisfy \( \mathcal{O}(3q) = \mathcal{O}(3p) \)?

3. For the line bundle \( L = \mathcal{O}_{\mathbb{C}/\Lambda}(1+\tau) \) from the previous two problem sheets, how might we try to go about constructing a section of \( L^* \) if we didn’t already know that this is impossible?

4. Show that if \( n > 1 \), \( U \subset \mathbb{C}^n \) is open, \( K \subset U \) is compact, and \( U \setminus K \) is connected, then every holomorphic function on \( U \setminus K \) extends to a holomorphic function on \( U \). Some hints:
• Extend the function smoothly from \( U \setminus K_\varepsilon \) to \( U \), where \( K_\varepsilon \) denotes the closed \( \varepsilon \)-tube around \( K \). Then the failure of this extension to be holomorphic is supported in \( K_\varepsilon \).

• Use a cohomology vanishing theorem “with estimates”: If \( \alpha \in C^\infty_0 (\mathbb{R}^{0,1} \mathbb{C}^n) \) and if \( \bar{\partial} \alpha = 0 \), then \( \alpha = \bar{\partial} \beta \) for some \( \beta \in C^\infty_0 (\mathbb{C}^n) \) such that \( \text{supp}(\beta) \) is contained in the union of \( \text{supp}(\alpha) \) with all the bounded connected components of the complement of \( \text{supp}(\alpha) \).

[Another good exercise is to think about why we cannot require that \( \text{supp}(\beta) \subset \text{supp}(\alpha) \).]

• Philosophically, this vanishing theorem holds because the trivial line bundle \( \mathbb{C}^n \times \mathbb{C} \) is positive. For an actual proof, define \( \beta \) bounded connected components of the complement of \( \text{supp}(\alpha) \).

Exercises for Lecture 6

1. Let \( L \) be a holomorphic line bundle over some complex manifold. Let \( h \) be a Hermitian metric on \( L \). Fix \( p \in M \) and a holomorphic chart \((z_1, ..., z_\text{n})\) based at \( p \) such that \( \omega_h|_p = \frac{1}{2} \sum \lambda_k dz^k \wedge d\bar{z}^k \) with \( \lambda_k \in \mathbb{R} \). Show that there exists a local holomorphic trivialization \( L \cong U \times \mathbb{C} \) near \( p \) such that \( h \) can be written as \( e^{-\varphi} |w|^2 \) with \( \varphi(z) = \frac{1}{2} \sum \lambda_k |z_k|^2 + O(|z|^3) \).

2. Let \( D \) denote the unit disk in \( \mathbb{C} \) with its standard coordinate \( z \). Let \( u : D \rightarrow \mathbb{C} \) be smooth. Show that \( \bar{\partial}^* (ud\bar{z}) = -u_z + \varphi_z u \) with respect to the Hermitian metric given by \( h = e^{-\varphi} |w|^2 \) on the trivial line bundle over \( D \) (with fiber coordinate \( w \)) and the usual flat metric on \( D \).

3. Some linear algebra of \((1,1)\)-forms on \( \mathbb{C}^n \).

• With notation as in class, verify the formulas \( g(u,v) = \omega(u,Jv) \) and \( g(\frac{\partial}{\partial z^j}, \frac{\partial}{\partial z^k}) = G_{jk} \).

• If \( \omega \) is positive and if \( W \subset \mathbb{C}^n \) is a complex subspace of complex dimension \( k \), then show that the restriction of the \( 2k \)-form \( \frac{\partial}{\partial z^j} \bar{\partial} \frac{\partial}{\partial z^k} \) to \( W \) is the volume form on \( W \) induced by the metric \( g \).

• If \( \omega \) is positive with associated inner product \( g \), and if \( \beta \) is another \((1,1)\)-form with associated symmetric bilinear form \( b \), then \( 2n\omega^{n-1} \wedge \beta = (\text{trace}_b)\omega^n \).

4. Let \( G \) denote the subgroup of \( \text{GL}(2, \mathbb{C}) \) generated by \( \begin{pmatrix} 0 & \lambda \noalign{
\hline} 1 & 0 \end{pmatrix} \). Show that the compact complex surface \( M = (\mathbb{C}^2 \setminus 0)/G \) is not a Kähler manifold. Hint: Compute \( H^2(M, \mathbb{R}) \).

Exercises for Lecture 7

1. Let \((L, h) \rightarrow M\) be a Hermitian holomorphic line bundle. Denote the dimension of the space of global holomorphic sections of \( L \) by \( N + 1 \). The precise definition of the pullback of “the Fubini-Study metric on \( \mathbb{P}^N \)” under \( \varphi_L \) is \( \varphi_L^* \omega_{\text{FS}} \equiv -i \bar{\partial} \partial \log(|s|^2/\sum |s_i|^2) \), where \( s \) is any nonvanishing local section of \( L \), and \( \{s_i\} \) is a chosen basis of global sections of \( L \). Compare \( \varphi_L^* \omega_{\text{FS}} \) and \( \omega_h \).

2. Show that the Hopf surface \((\mathbb{C}^2 \setminus 0)/\langle \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rangle \) satisfies the global \( \partial \bar{\partial} \)-lemma.

3. Check by computation that the Fubini-Study metric \( \omega_{\text{FS}} = i \bar{\partial} \partial \log(1 + |z_1|^2 + \cdots + |z_n|^2) \) on \( \mathbb{C}^n \subset \mathbb{P}^n \) is a Kähler-Einstein metric, i.e. that \( \text{Ric}(g_{\text{FS}}) = \lambda g_{\text{FS}} \) for some \( \lambda \in \mathbb{R} \).

4. Compute the linearization of the Monge-Ampère equation \((\omega + i \bar{\partial} \partial u)^n = e^F \omega^n \) at \( u = 0 \). It is helpful to try this on \( \mathbb{C}^n \) first, where the equation can be rewritten as \( \text{det}(\delta_{jk} + \frac{\partial^2 u}{\partial z^j \partial z^k}) = e^F \).

Exercises for Lecture 8

1. Prove that \( K_{\mathbb{P}^n} \cong \mathcal{O}_{\mathbb{P}^n}(-n-1) \) by checking that if \( F \) is a homogeneous polynomial of degree \( n + 1 \) on \( \mathbb{C}^{n+1} \), then the meromorphic sections of \( K_{\mathbb{P}^n}|_{U_i} \) defined by

\[
( -1 )^s \frac{dz_0}{z_1} \wedge \ldots \wedge \frac{d\bar{z}_{i-1}}{z_i} \wedge \frac{d\bar{z}_{i+1}}{z_i} \wedge \ldots \wedge \frac{d\bar{z}_n}{z_i} \bigg|_{z_i^{-(n+1)}F(z_0, \ldots, z_n)}
\]

patch up as a global meromorphic section of \( K_{\mathbb{P}^n} \) with simple poles along \( F = 0 \).
(2) Show that the total space of the canonical bundle of a complex manifold has trivial canonical bundle, by constructing a nowhere vanishing top degree holomorphic form on the total space.

(3) Show that a torus invariant function $\varphi = \varphi(|z|, |w|)$ on $\mathbb{C}^2$ satisfies $\det(\varphi_{jk}) = 1$ if and only if the function $u(x, y) = \varphi(e^x, e^y)$ on $\mathbb{R}^2$ satisfies $u_{xx}u_{yy} - u_{xy}^2 = 16e^{2x+2y}$.

(4) Consider the smooth complex 3-manifold $H \subset \mathbb{C}^2 \times \mathbb{P}^1$ cut out by the system of equations $(x \ z)(u \ y) = (0 \ 0)$. How does the map $H \to \mathbb{C}$ given by $z - w$ relate to the discussion in class?

(5) Prove that a smooth algebraic curve of degree $d$ in $\mathbb{P}^2$ has genus $\frac{1}{2}(d-1)(d-2)$ (in the sense of surface topology) by deforming the curve to a union of $d$ straight lines in general position.

**Exercises for Lecture 9/10**

(1) Compute the moment map of LeBrun’s space $(\mathbb{C}^2, \omega)$, $\omega = i\partial \bar{\partial} \varphi$, $\varphi = u^2 + v^2 + m(u^4 + v^4)$, with respect to the holomorphic isometric circle action $\zeta_1(z_1, z_2) = (\zeta_1, \zeta_2)$. (By definition, this is the unique function $\mu : \mathbb{C}^2 \to \mathbb{R}$ satisfying $\mu(0) = 0$ and $d\mu = X \cdot \omega$ for the infinitesimal generator $X$ of the circle action. Hint: $X \cdot \omega = X \cdot d\varphi = L_X(d\varphi) - d(X \cdot d\varphi)$.)

(2) Use integration by parts to show that if $u_1$ and $u_2$ satisfy $(\omega + dd^c u) = e^f \omega^2$ $(i = 1, 2)$ on a compact Kähler manifold, then $u_1 - u_2$ is a constant.

(3) Demystify the integration-by-parts trick for the complex Monge-Ampère equation by using a cofactor expansion to show that if $u : \mathbb{R}^n \to \mathbb{R}$ is a smooth function, then $\det \text{Hess} u$ can be written as the divergence of a vector field depending on first and second derivatives of $u$.

(4) Let $M^n$ be a compact Riemannian manifold. Let $E \to M$ be a Hermitian vector bundle with a Hermitian connection. Let $A$ be a smooth section of the bundle $\text{End} E$.

- Use Moser iteration to show that if some smooth section $\sigma$ of $E$ satisfies the Schrödinger type equation $\nabla^* \nabla \sigma + A \sigma = 0$, then $\|\sigma\|_\infty \leq C(M, E)(1 + \|A\|_\infty^{n/4})\|\sigma\|_2$.

- Fix a finite-dimensional subspace $V$ of the space of solutions to $\nabla^* \nabla \sigma + A \sigma = 0$. Pick an $L^2$-orthonormal basis $\{\sigma_i\}$ of $V$. Show that the function $\rho : M \to [0, \infty)$ given by $\rho(x) = \frac{1}{2} \sum_i |\sigma_i(x)|^2$ is independent of the choice of such a basis $V$.

- Show that $\rho(x) = \sup \{|\sigma(x)|^2 : \sigma \in V, \|\sigma\|_2 = 1\}$.

- Deduce that $\dim V = \int_M \rho(x) \, d\text{vol}(x) \leq C(M, E)(1 + \|A\|_\infty^{n/2})$, independent of $V$.

- Study some special cases: eigenfunctions of the Laplacian, holomorphic sections of $L^k$, ... .