Exercises.

(1) (Get a feel for $\simeq$-equivalence) Order these functions with respect to $\preceq$. Determine which, if any, are $\simeq$-equivalent.

$n^23^n$, $4^n$, $2^{2n}$, $n^2(\log n)^2$, $n^{2n^2+2}$, $n^{10} + 10^n$, $\sqrt{n}3\sqrt{n}$, $f(n)$ (defined as below)

Define $f(n) = \exp^n(e)$, where $\exp(n) = e^n$.

(2) (This example shows why we need the linear term in the definition of $\simeq$-equivalence.) Show that the Dehn function of $\langle a \mid \emptyset \rangle$ is $\delta(n) \equiv 0$, and the Dehn function of $\langle a, b \mid b \rangle$ is $\delta(n) = n$.

This is Exercise 1.3.1 in Bridson’s The geometry of the word problem.

(3) Show that $\delta_{\mathbb{Z}^m}(n) \simeq n^2$ for $m > 1$.

(4) The Heisenberg group is the group with presentation

$H = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle$.

Use the fact that every element can be expressed in a unique way as a word of the form $a^pb^qc^r$ for some integers $p, q, r$ to show that $\delta_H(n) \preceq n^3$.

This is Exercise 2.14 in Riley’s What is a Dehn function?

(5) (a) Prove that if $P_1$ and $P_2$ are two different finite presentations for a group $G$, then $\delta_{P_1}(n) \simeq \delta_{P_2}(n)$.

(Use the fact that any two presentations of $G$ are related by a sequence of Tietze moves.)

(b) If you are familiar with quasi-isometries, sketch a proof that the Dehn function of a group is a quasi-isometry invariant. (The idea is straightforward, but the details get technical.)

(6) Show that if $G = \langle S \mid R \rangle$ is finitely presented, and $w \in F(S)$ with $w =_G 1$ and $|w| = n$, then there is an expression

$$\prod_{i=1}^{m} u_j r_j u_i^{-1}$$

with $|u_i| \leq |w| + C\delta_G(n)$, where $C$ is the maximum of the lengths of the relators in $R$.

References.


(1) Finish the proof from the lecture, that hyperbolic groups have a Dehn presentation.

(2) The group $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ is a Baumslag-Solitar group. Sketch a picture of the universal cover of its presentation complex. This will be useful tomorrow.

(3) Show that $\mathbb{Z}^n$ has a synchronous 2-fellow traveling combing.

(4) (Asynchronous combings) Define a reparametrization to be a function $\rho : \mathbb{N} \to \mathbb{N}$ with $\rho(0) = 0$ and $\rho(n) \in \{\rho(n-1), \rho(n-1) + 1\}$ for all $n \geq 1$.

A normal form $\sigma$ for $G$ with generating set $S$ asynchronously $k$-fellow travels if for all $g, h \in G$ with $d_G(g, h) = 1$, there exist reparametrizations $\rho, \rho'$ such that $d_S(\sigma_g(\rho(t)), \sigma_h(\rho'(t))) \leq k$. Now $(G, S)$ is asynchronously combable if there exists an asynchronous $k$-fellow traveling normal form for $G$. As in the lecture, define the length function by

$$L(n) = \max\{\ell(\sigma_g) \mid d_G(g, e) = 1\}$$

Show that if $(G, S)$ has an asynchronous $k$-fellow traveling combing then

$$\delta_G(n) \leq n L(n)$$

by modifying the proof from the lecture.

The Baumslag-Solitar group $BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ is asynchronously combable with $L(n) \simeq 2^n$. See Riley’s Filling Functions for a description of the normal form that works.

(5) (Exercise 3.4.1 from Bridson’s The geometry of the word problem)

(a) A subgroup $H$ of a group $G$ is a retract if there is a homomorphism $G \to H$ whose restriction to $H$ is the identity. Show that if $H$ is a retract of a finitely presented group $G$, then $H$ is finitely presented, and $\delta_H(n) \leq \delta_G(n)$.

(Hint: First note that $H$ is finitely generated. Show that a generating set for $H$ can be extended to one for $G$, and use this to produce a finite presentation for $G$.)

(b) Let $G_1$ and $G_2$ be infinite, finitely presented groups. Show that

$$\delta_{G_1 \times G_2}(n) \simeq \max\{n^2, \delta_{G_1}(n), \delta_{G_2}(n)\}$$

and

$$\delta_{G_1 \ast G_2}(n) \simeq \max\{\delta_{G_1}(n), \delta_{G_2}(n)\}.$$
DEHN FUNCTIONS
EXERCISES FOR DAY 3 (WEDNESDAY, JUNE 17)

(1) Show that if $H$ is an HNN extension of $G$ defined by an injective homomorphism $\phi : G \to B < G$ then there is an exponential upper bound on the Dehn function of $G$.

(2) Finish the proof that the Dehn function of $BS(1, 2)$ is $2^n$.

(3) $G = \langle a, ts \mid tat^{-1} = a^2, sts^{-1} = t^2 \rangle$. Show that the Dehn function is $2^{2^n}$.

(4) Define analogous groups (to the one define in the previous problem) with Dehn functions equal to any tower of exponential functions.

(5) Show that the group presented by $G = \langle a, t, s \mid tat^{-1} = a^2, sas^{-1} = t \rangle$ (Gersten’s group) has Dehn function growing faster than any tower of exponentials.
(1) Complete the details of the proof that the Dehn function of the Heisenberg group is $n^3$.

(2) Complete the details of the proof that the Dehn function of the lattice in Sol is exponential.

(3) Given a pair of groups $\Gamma < G$ with word metrics $d_H$ and $d_G$, the distortion of $\Gamma$ in $G$ is

$$\text{disto}^G_\Gamma(n) = \max\{d_G(e, g) \mid g \in \Gamma; d_H(e, h) \leq n\}$$

(a) If $H = \langle a, b, c \mid [a, b] = c, [a, c] = [b, c] = 1 \rangle$ is the Heisenberg group, what is $\text{disto}^H_\langle c \rangle(n)$?

(b) If $G = BS(1, 2) = \langle a, t \mid tat^{-1} = a^2 \rangle$ what is $\text{disto}^G_\langle a \rangle(n)$?

(c) Define $\phi : F(a, b, c) \to F(a, b, c)$ by $a \mapsto ab$, $b \mapsto bc$, and $c \mapsto c$. Now let $G_t$ be the HNN extension $G_t = \langle a, b, c, t \mid txt^{-1} = \phi(x) \text{ for } x = a, b, c \rangle$

(i) What is the distortion of $F(a, b, c)$ in $G_t$?

(ii) Define $G_s$ analogously to $G_t$, with $s$ taking the place of $t$. What can you say about the Dehn function of the group $G_t * F(a, b, c) G_s$?

(This group is sometimes called the double of $G_t$ over $F(a, b, c)$.

(4) CAT(0) groups are groups which act properly discontinuously and cocompactly by isometries on a CAT(0) space. To find combings, one only needs an important consequence of the CAT(0) condition, namely the fact that the metric on a CAT(0) space $X$ is convex. This means that given a pair of geodesics $\alpha, \beta : [0, 1] \to X$ parametrized proportional to arclength, one has

$$d_X(\alpha(t), \beta(t)) \leq td_X(\alpha(0), \beta(0)) + (1 - t)d_X(\alpha(1), \beta(1))$$

Show that if $X$ has a convex metric, and $G$ acts on $X$ properly discontinuously and cocompactly by isometries, then $G$ is synchronously combable, with linear length function.

Thus any such $G$ satisfies a quadratic isoperimetric inequality.
(1) Let $G$ be a finitely presented group. Show that the word problem in $G$ is solvable if and only if the Dehn function of $G$ is recursive.

(Take recursive to mean that there is an algorithm which, given $n$, computes $f(n)$ in a finite amount of time.)

(2) Finish the proof of Groft’s theorem. Let $X$ be the universal cover of a $K(G, 1)$ with finite $(k+1)$-skeleton.

(a) Let $k \geq 2$. Show that given a $k$-sphere $\sigma : S^k \to X^{(k)}$, and a filling of the corresponding homology class $\hat{\sigma}$ by a $(k+1)$-chain $\beta$, there is an extension $\tilde{\sigma} : D^{k+1} \to X^{k+1}$ such that $\text{Vol}(\tilde{\sigma}) = \text{mass}(\beta)$.

This shows that for a given sphere, you can do no better by using chains to fill than by using just disks.

(This was worked out in the problem session.)

(b) Let $k \geq 3$. Show that given any cycle $\alpha \in Z_k(X, \mathbb{Z})$, there exists a $k$-sphere $\sigma : S^k \to X^{(k)}$ such that

$$\text{Vol}(\sigma) = \text{mass}(\alpha) \text{ and } F\text{Vol}(\sigma) = F\text{V}(\alpha)$$

(c) Explain why the above statements fail when $k$ is lower than the threshold mentioned.

(d) Use the statements in (1) and (2) to finish the proof of Groft’s Theorem.