Problems for mapping class groups, day 1

1. Carefully prove that $\hat{i}(a,b)$ depends only on the homology classes of the curves a and b in the surface S.

2. Over all possible pairs of homotopy classes of curves a, b over all possible surfaces, exactly what values of pairs $(i(a,b), \hat{i}(a,b))$ are possible?

3. Verify that composition in $\mathrm{Mod}(S)$ is well defined.

4. Show that $\mathrm{Homeo}_0(S, \partial S)$, the path component of the identity in $\mathrm{Homeo}^+(S, \partial S)$, is a normal subgroup.

5. Carefully show that the following are equivalent for $\phi, \psi \in \mathrm{Homeo}^+(S, \partial S)$, quoting theorems from the lecture:

   - ϕ and ψ are in the same path component of $\mathrm{Homeo}^+(S, \partial S)$ in the compact-open topology.
   - ϕ and ψ are in the same coset of $\mathrm{Homeo}_0(S, \partial S)$ the path component of the identity.
   - ϕ and ψ are homotopic.

6. What happens if we do not demand that homeomorphisms fix ∂S pointwise or that homotopies are relative to ∂S in the definition of $\mathrm{Mod}(S)$?

7. Show that every finite subgroup of the isometry group of S^2 embeds in $\mathrm{Mod}(S)$ for some surface S.

8. An arc on a surface S is a map of an interval to S. We demand that endpoints of intervals map to ∂S, and if an endpoint is missing, then the map is proper (so the missing "endpoint" maps to a puncture of the surface). Arcs are considered up to proper homotopy relative to endpoints. State and prove a bigon criterion for arcs, and for an arc and a curve.

9. State and prove a criterion for checking whether a single curve has the minimal number of self-intersections for its homotopy class.

10. Classify the finite-order elements of $\mathrm{SL}_2(\mathbb{Z})$. Conclude that a surface can have an interesting mapping class group, but still have limited possibilities for orders of finite-order elements.

11. Verify that for any mapping class f and any Dehn twist T_a, we have $T_{f(a)} = f T_a f^{-1}$.

12. Verify the braid relation to yourself by drawing pictures. The braid relation states that if a, b are homotopy classes of curves on S and $i(a,b) = 1$, then $T_a T_b T_a = T_b T_a T_b$.

13. Suppose a curve a is not nullhomotopic and is not homotopic to a puncture. Show that T_a is nontrivial in $\mathrm{Mod}(S)$.

14. Draw pictures of curves α and β in a surface S with $i(\alpha, \beta) \neq 0$ and with T_a, T_b nontrivial. Verify that $[T_a, T_b] \neq 1$ by drawing pictures.

15. Give a factorization of a finite order element of a mapping class group as a product of Dehn twists.

16. Let $\mathrm{Mod}^{\pm}(S)$ be the version of the mapping class group that includes classes of orientation-reversing homeomorphisms, up to equivalence by isotopy. Show that $\mathrm{Mod}^{\pm}(S)$ is a semidirect product of $\mathbb{Z}/2\mathbb{Z}$ acting on $\mathrm{Mod}(S)$. Describe the action of $\mathbb{Z}/2\mathbb{Z}$ on the set of Dehn twists coming from some such decomposition.

Problems for mapping class groups, day 2

Now that you've heard about the Alexander method, you may want to give some problems from yesterday another try. New problems for today:

1. What conditions must a set of Dehn twists satisfy in order for the intersection of their centralizers to be trivial?
2. Show that the center of every finite-index subgroup of $\mathrm{Mod}(S_g)$ is trivial for $g \geq 3$.
3. Find a nontrivial central element of $\mathrm{Mod}(S_2)$.
4. Look up the lantern relation and verify it using the Alexander method.
5. Use the lantern relation, the change of coordinates principle, and the Dehn–Lickorish theorem to prove that $\mathrm{Mod}(S_g)$ has trivial abelianization for $n \geq 3$.
6. Find a specific finite generating set for $\mathrm{Mod}(S_2)$. (Follow the proof of the Dehn–Lickorish theorem.)
7. Find a path of length 3 in the curve complex of a surface.
8. Pick a surface S (this is interesting for $S = S_2$). Pick a mapping class f on that is not a product of Dehn twists in an obvious way. Pick nonseparating simple closed curves a and b with $i(a, b) = 1$.
   (a) Find paths between a and $f(a)$ in the curve complex, the nonseparating curve complex, and the modified nonseparating curve complex.
   (b) Express f as a product in terms of mapping classes stabilizing a and the Dehn twist T_b.
   (c) This is probably hard, depending on your example: express f as a product of Dehn twists.

Problems for mapping class groups, day 3

Please try problem 5 from yesterday: show that $H_1(\mathrm{Mod}(S_g);\mathbb{Z})$ is trivial if $g \geq 3$. New problems for today:

1. Assuming that $\mathrm{Mod}(S_g)$ is finitely presentable, deduce that $\mathrm{Mod}(S_{g,n}^b)$ is finitely presentable (n punctures, b boundary components).

2. Prove the Euler–Poincaré formula.

3. Try out Thurston's example: Model $S_{0,4}$ as the plane with three punctures p_1, p_2, p_3 in a row on the x-axis (the fourth puncture is the point at infinity). Let σ_1 be the counterclockwise half-twist switching p_1 and p_2, and let σ_2 be the counterclockwise half-twist switching p_2 and p_3. Let $f = \sigma_1^{-1}\sigma_2$. Pick an essential simple closed curve c and draw $f^k(c)$ for several positive integers k.

   Here's how to interpret your picture: roughly speaking, the pieces of your curve are somehow approaching the leaves of an unstable singular foliation for f. This picture led Thurston to the idea of *train tracks*, one of three constructions commonly used to characterize the structure of pseudo-Anosov mapping classes.

4. Recall that we identify the mapping class group $\mathrm{Mod}(\mathbb{T}^2)$ of the torus $\mathbb{T}^2$ with $\mathrm{SL}(2,\mathbb{Z})$ via the action on the fundamental group. Consider the mapping class f corresponding to the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Verify that this mapping class is Anosov:

   (a) Concretely describe the transverse measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ that some representative of f leaves invariant (up to scaling).

   (b) Find the representative ϕ of f that respects these foliations.

   (c) Find the stretch factor λ.

   (d) Verify that ϕ sends leaves of $\mathcal{F}^s$ to leaves of $\mathcal{F}^s$ (resp. for $\mathcal{F}^u$).

   (e) Verify that ϕ sends μ_s to $\lambda^{-1}\mu_s$ (resp. μ_u to $\lambda\mu_u$) by plugging in an arbitrary transverse arc.

5. Build a mapping class in $\mathrm{Mod}(S_2)$ that is pseudo-Anosov by taking a product of Dehn twists. Use the Nielsen–Thurston classification to deduce that your mapping class really is pseudo-Anosov.

Problems for mapping class groups and $\mathrm{Out}(F_n)$, day 4

Distinguishing between boundary components and punctures is really important today, so here's a reminder that S_g^1 has a boundary component, and $S_{g,1}$ has a puncture.

New problems for today:

1. Show that the map $\mathrm{Mod}(S_g) \to \mathrm{Out}(\pi_1(S_g))$ is well defined. (Hint: consider the point pushing map $\pi_1(S_g) \to \mathrm{Mod}(S_{g,1})$ and the action of $\mathrm{Mod}(S_{g,1})$ on $\pi_1(S_g)$.)

2. Derive the Dehn–Nielsen–Baer theorem for S_g^1 from the Dehn–Nielsen–Baer theorem for S_g.

3. Let a be a separating simple closed curve in S_2^1. We can relate S_g^1 to $S_{g,1}$ by capping the boundary with a puncture, and to S_g by forgetting the puncture, so we can also consider a to be curve in $S_{g,1}$ and S_g. A

   Compute the action of $T_a \in \mathrm{Mod}(S_g^1)$ on $\pi_1(S_g^1)$ in terms of a standard basis for $\pi_1(S_g^1)$. Use your answer to also give the action of $T_a \in \mathrm{Mod}(S_{g,1})$ on $\pi_1(S_g)$ in terms of a standard generating set, and give the image of $T_a \in \mathrm{Mod}(S_g)$ under the map to $\mathrm{Out}(\pi_1(S_g)$.

4. Use folding to check which of the following homomorphisms $F_2 \to F_2$ is an automorphism and which isn't:

$$f_1\colon \begin{array}{ccc} x & \mapsto & xy^2xyxy \\ y & \mapsto & yxyxy, \end{array} \quad \text{or} \quad f_2\colon \begin{array}{ccc} x & \mapsto & xyxy^2xyxy \\ y & \mapsto & yxyxy \end{array}$$

5. Express the following automorphism of F_3 as a product of Nielsen moves and compute its inverse. (It may be easier to do this with Nielsen reduction than with folding.)

$$f\colon \begin{array}{ccc} x & \mapsto & xyz \\ y & \mapsto & xyzx^{-1} \\ y & \mapsto & zyzx^{-1} \end{array}$$

6. Show that $\mathrm{Aut}(F_n)$ has a generating set consisting of $4(n-1)$ transvections and a single inversion.

7. Find a free abelian subgroup of $\mathrm{Out}(F_n)$ of rank $2n-3$.

Problems for mapping class groups and $\mathrm{Out}(F_n)$, day 5

New problems for today:

1. Use the symplectic form
$$J = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$
and work in the group $\mathrm{Sp}(4,\mathbb{Z}) = \{A \in \mathrm{GL}(n,\mathbb{Z}) | A^T J A = J\}$. Suppose $A \in \mathrm{Sp}(4,\mathbb{Z})$ is a "row operation" matrix such that for any B, the last row of AB is the sum of the first and last row of B. Then A has the form:
$$A = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$
What is the simplest way we can fill in the rest of A and still get a matrix in $\mathrm{Sp}(4,\mathbb{Z})$? (Is there a "symplectic row operation" that differs from the standard row operation described above by changing only one matrix entry?)

2. Repeat the previous problem, but with other row operations.

3. Find a mapping class in $\mathrm{Mod}(S_2)$ that maps to the matrix in your answer to problem 1. (This one's tricky; the answer is in Farb–Margalit.)

4. What is the action of a Dehn twist on $H_1(S)$?

5. Give an example of a relation that is true for elementary matrices in $\mathrm{GL}(n,\mathbb{Z})$ but is not true for the corresponding Nielsen moves. Use this to give an example of an element of IA_n.

6. Suppose A and B are abelian groups and we have a central extension
$$1 \to A \to G \xrightarrow{\pi} B \to 1$$
with $A = [G,G]$. Show that the kernel of the natural map $\ker(\mathrm{Aut}(G) \to \mathrm{Aut}(B))$ is isomorphic to the abelian group $\mathrm{Hom}(B,A)$. (Hint: if $\phi\colon B \to A$ is a homomorphism, then $\psi\colon G \to G$ defined by $\psi(g) = \phi(\pi(g))g$ is an automorphism.)

7. Suppose a and b are a pair of disjoint simple closed curves in S_3^1 that separate the surface into two genus-one surfaces. Let f be the bounding pair map $T_a T_b^{-1}$. Check that f is in the Torelli group $\mathcal{I}_g$. Compute the Johnson homomorphism of f.

8. What are the minimal graphs of rank 3 with the maximal number of vertices? What are the faces of the simplices corresponding to each one in the simplicial structure on Outer space X_3?