Problems for mapping class groups, day 1

1. Carefully prove that \( \hat{i}(a,b) \) depends only on the homology classes of the curves \( a \) and \( b \) in the surface \( S \).

2. Over all possible pairs of homotopy classes of curves \( a, b \) over all possible surfaces, exactly what values of pairs \( (i(a,b), \hat{i}(a,b)) \) are possible?

3. Verify that composition in \( \text{Mod}(S) \) is well defined.

4. Show that \( \text{Homeo}_0(S, \partial S) \), the path component of the identity in \( \text{Homeo}^+(S, \partial S) \), is a normal subgroup.

5. Carefully show that the following are equivalent for \( \phi, \psi \in \text{Homeo}^+(S, \partial S) \), quoting theorems from the lecture:
   - \( \phi \) and \( \psi \) are in the same path component of \( \text{Homeo}^+(S, \partial S) \) in the compact-open topology.
   - \( \phi \) and \( \psi \) are in the same coset of \( \text{Homeo}_0(S, \partial S) \) the path component of the identity.
   - \( \phi \) and \( \psi \) are homotopic.

6. What happens if we do not demand that homeomorphisms fix \( \partial S \) pointwise or that homotopies are relative to \( \partial S \) in the definition of \( \text{Mod}(S) \)?

7. Show that every finite subgroup of the isometry group of \( S^2 \) embeds in \( \text{Mod}(S) \) for some surface \( S \).

8. An arc on a surface \( S \) is a map of an interval to \( S \). We demand that endpoints of intervals map to \( \partial S \), and if an endpoint is missing, then the map is proper (so the missing "endpoint" maps to a puncture of the surface). Arcs are considered up to proper homotopy relative to endpoints. State and prove a bigon criterion for arcs, and for an arc and a curve.

9. State and prove a criterion for checking whether a single curve has the minimal number of self-intersections for its homotopy class.

10. Classify the finite-order elements of \( \text{SL}_2(\mathbb{Z}) \). Conclude that a surface can have an interesting mapping class group, but still have limited possibilities for orders of finite-order elements.

11. Verify that for any mapping class \( f \) and any Dehn twist \( T_a \), we have \( T_f(a) = fT_af^{-1} \).

12. Verify the braid relation to yourself by drawing pictures. The braid relation states that if \( a, b \) are homotopy classes of curves on \( S \) and \( i(a,b) = 1 \), then \( T_aT_bT_a = T_bT_aT_b \).

13. Suppose a curve \( a \) is not nullhomotopic and is not homotopic to a puncture. Show that \( T_a \) is nontrivial in \( \text{Mod}(S) \).

14. Draw pictures of curves \( \alpha \) and \( \beta \) in a surface \( S \) with \( i(\alpha, \beta) \neq 0 \) and with \( T_a, T_b \) nontrivial. Verify that \( [T_a, T_b] \neq 1 \) by drawing pictures.

15. Give a factorization of a finite order element of a mapping class group as a product of Dehn twists.

16. Let \( \text{Mod}^\pm(S) \) be the version of the mapping class group that includes classes of orientation-reversing homeomorphisms, up to equivalence by isotopy. Show that \( \text{Mod}^\pm(S) \) is a semidirect product of \( \mathbb{Z}/2\mathbb{Z} \) acting on \( \text{Mod}(S) \). Describe the action of \( \mathbb{Z}/2\mathbb{Z} \) on the set of Dehn twists coming from some such decomposition.
Problems for mapping class groups, day 2

Now that you’ve heard about the Alexander method, you may want to give some problems from yesterday another try. New problems for today:

1. What conditions must a set of Dehn twists satisfy in order for the intersection of their centralizers to be trivial?

2. Show that the center of every finite-index subgroup of $\text{Mod}(S_g)$ is trivial for $g \geq 3$.

3. Find a nontrivial central element of $\text{Mod}(S_2)$.

4. Look up the lantern relation and verify it using the Alexander method.

5. Use the lantern relation, the change of coordinates principle, and the Dehn–Lickorish theorem to prove that $\text{Mod}(S_g)$ has trivial abelianization for $n \geq 3$.

6. Find a specific finite generating set for $\text{Mod}(S_2)$. (Follow the proof of the Dehn–Lickorish theorem.)

7. Find a path of length 3 in the curve complex of a surface.

8. Pick a surface $S$ (this is interesting for $S = S_2$). Pick a mapping class $f$ on that is not a product of Dehn twists in an obvious way. Pick nonseparating simple closed curves $a$ and $b$ with $i(a, b) = 1$.

   (a) Find paths between $a$ and $f(a)$ in the curve complex, the nonseparating curve complex, and the modified nonseparating curve complex.

   (b) Express $f$ as a product in terms of mapping classes stabilizing $a$ and the Dehn twist $T_b$.

   (c) This is probably hard, depending on your example: express $f$ as a product of Dehn twists.
Problems for mapping class groups, day 3

Please try problem 5 from yesterday: show that $H_1(\text{Mod}(S_g);\mathbb{Z})$ is trivial if $g \geq 3$. New problems for today:

1. Assuming that $\text{Mod}(S_g)$ is finitely presentable, deduce that $\text{Mod}(S_{g,n}^b)$ is finitely presentable ($n$ punctures, $b$ boundary components).

2. Prove the Euler–Poincaré formula.

3. Try out Thurston’s example: Model $S_{0,4}$ as the plane with three punctures $p_1, p_2, p_3$ in a row on the $x$-axis (the fourth puncture is the point at infinity). Let $\sigma_1$ be the counterclockwise half-twist swapping $p_1$ and $p_2$, and let $\sigma_2$ be the counterclockwise half-twist swapping $p_2$ and $p_3$. Let $f = \sigma_1^{-1}\sigma_2$. Pick an essential simple closed curve $c$ and draw $f^k(c)$ for several positive integers $k$.

Here’s how to interpret your picture: roughly speaking, the pieces of your curve are somehow approaching the leaves of an unstable singular foliation for $f$. This picture led Thurston to the idea of train tracks, one of three constructions commonly used to characterize the structure of pseudo-Anosov mapping classes.

4. Recall that we identify the mapping class group $\text{Mod}(\mathbb{T}^2)$ of the torus $\mathbb{T}^2$ with $\text{SL}(2,\mathbb{Z})$ via the action on the fundamental group. Consider the mapping class $f$ corresponding to the matrix \[
\begin{pmatrix}
2 & 1 \\
1 & 1
\end{pmatrix}
\]. Verify that this mapping class is Anosov:
   
   (a) Concretely describe the transverse measured foliations $\mathcal{F}^s$, $\mathcal{F}^u$ that some representative of $f$ leaves invariant (up to scaling).
   
   (b) Find the representative $\phi$ of $f$ that respects these foliations.
   
   (c) Find the stretch factor $\lambda$.
   
   (d) Verify that $\phi$ sends leaves of $\mathcal{F}^s$ to leaves of $\mathcal{F}^s$ (resp. for $\mathcal{F}^u$).
   
   (e) Verify that $\phi$ sends $\mu_s$ to $\lambda^{-1}\mu_s$ (resp. $\mu_u$ to $\lambda\mu_u$) by plugging in an arbitrary transverse arc.

5. Build a mapping class in $\text{Mod}(S_2)$ that is pseudo-Anosov by taking a product of Dehn twists. Use the Nielsen–Thurston classification to deduce that your mapping class really is pseudo-Anosov.
Problems for mapping class groups and Out($F_n$), day 4

Distinguishing between boundary components and punctures is really important today, so here’s a reminder that $S^1_g$ has a boundary component, and $S_{g,1}$ has a puncture.

New problems for today:

1. Show that the map $\text{Mod}(S_g) \to \text{Out}(\pi_1(S_g))$ is well defined. (Hint: consider the point pushing map $\pi_1(S_g) \to \text{Mod}(S_{g,1})$ and the action of $\text{Mod}(S_{g,1})$ on $\pi_1(S_g)$.)

2. Derive the Dehn–Nielsen–Baer theorem for $S^1_g$ from the Dehn–Nielsen–Baer theorem for $S_g$.

3. Let $a$ be a separating simple closed curve in $S^1_2$. We can relate $S^1_g$ to $S_{g,1}$ by capping the boundary with a puncture, and to $S_g$ by forgetting the puncture, so we can also consider $a$ to be curve in $S_{g,1}$ and $S_g$. A

Compute the action of $T_a \in \text{Mod}(S^1_g)$ on $\pi_1(S^1_g)$ in terms of a standard basis for $\pi_1(S^1_g)$. Use your answer to also give the action of $T_a \in \text{Mod}(S_{g,1})$ on $\pi_1(S_g)$ in terms of a standard generating set, and give the image of $T_a \in \text{Mod}(S_g)$ under the map to Out($\pi_1(S_g)$).

4. Use folding to check which of the following homomorphisms $F_2 \to F_2$ is an automorphism and which isn’t:

\[
\begin{align*}
&f_1: & x & \mapsto xy^2xyxy & \quad \text{or} \quad f_2: & x & \mapsto xyxy^2xyxy \\
& & y & \mapsto yxyxy & & y & \mapsto yxyxy
\end{align*}
\]

5. Express the following automorphism of $F_3$ as a product of Nielsen moves and compute its inverse. (It may be easier to do this with Nielsen reduction than with folding.)

\[
\begin{align*}
&f: & x & \mapsto xyz \\
& & y & \mapsto xyxz^{-1} \\
& & y & \mapsto yzx^{-1}
\end{align*}
\]

6. Show that $\text{Aut}(F_n)$ has a generating set consisting of $4(n-1)$ transvections and a single inversion.

7. Find a free abelian subgroup of Out($F_n$) of rank $2n-3$. 
Problems for mapping class groups and $\text{Out}(F_n)$, day 5

New problems for today:

1. Use the symplectic form
   \[
   J = \begin{pmatrix}
   0 & 1 & 0 & 0 \\
   -1 & 0 & 0 & 0 \\
   0 & 0 & 0 & 1 \\
   0 & 0 & -1 & 0 
   \end{pmatrix}
   \]
   and work in the group $\text{Sp}(4, \mathbb{Z}) = \{ A \in \text{GL}(n, \mathbb{Z}) | A^T J A = J \}$. Suppose $A \in \text{Sp}(4, \mathbb{Z})$ is a “row operation” matrix such that for any $B$, the last row of $AB$ is the sum of the first and last row of $B$. Then $A$ has the form:
   \[
   A = \begin{pmatrix}
   * & * & * & * \\
   * & * & * & * \\
   * & * & * & * \\
   1 & 0 & 0 & 1 
   \end{pmatrix}
   \]
   What is the simplest way we can fill in the rest of $A$ and still get a matrix in $\text{Sp}(4, \mathbb{Z})$? (Is there a “symplectic row operation” that differs from the standard row operation described above by changing only one matrix entry?)

2. Repeat the previous problem, but with other row operations.

3. Find a mapping class in $\text{Mod}(S_2)$ that maps to the matrix in your answer to problem 1. (This one’s tricky; the answer is in Farb–Margalit.)

4. What is the action of a Dehn twist on $H_1(S)$?

5. Give an example of a relation that is true for elementary matrices in $\text{GL}(n, \mathbb{Z})$ but is not true for the corresponding Nielsen moves. Use this to give an example of an element of $\text{IA}_n$.

6. Suppose $A$ and $B$ are abelian groups and we have a central extension
   \[
   1 \to A \to G \xrightarrow{\pi} B \to 1
   \]
   with $A = [G, G]$. Show that the kernel of the natural map $\ker(\text{Aut}(G) \to \text{Aut}(B))$ is isomorphic to the abelian group $\text{Hom}(B, A)$. (Hint: if $\phi : B \to A$ is a homomorphism, then $\psi : G \to G$ defined by $\psi(g) = \phi(\pi(g))g$ is an automorphism.)

7. Suppose $a$ and $b$ are a pair of disjoint simple closed curves in $S_1^3$ that separate the surface into two genus-one surfaces. Let $f$ be the bounding pair map $T_a T_b^{-1}$. Check that $f$ is in the Torelli group $\mathcal{T}_g$. Compute the Johnson homomorphism of $f$.

8. What are the minimal graphs of rank 3 with the maximal number of vertices? What are the faces of the simplices corresponding to each one in the simplicial structure on Outer space $X_3$?