Exercise Sheet 3

1. Let $f, g, h: \mathbb{N}^+ \rightarrow \mathbb{U}$ be multiplicative functions and $x, y \in \mathbb{R}$. Recall the definition

$$
\mathbb{D}(f, g; y, x)^2 = \sum_{y < p \leq x} \frac{1 - \Re(f(p)g(p))}{p}.
$$

Check that $\mathbb{D}$ satisfies the triangle inequality

$$
\mathbb{D}(f, h; y, x) \leq \mathbb{D}(f, g; y, x) + \mathbb{D}(g, h; y, x).
$$

2. Recall from the discussion of the Selberg sieve that the main term of the upper bound to $\pi(x + y) - \pi(x)$ of the sieve was $\frac{y}{L}$ where

$$
L = \sum_{r \leq z \atop p|r \Rightarrow p \leq R} \frac{\mu(r)^2}{\varphi(r)}
$$

where $z = \sqrt{y}$ and $R \leq z$ is a fixed power of $z$. Also, the sieve weights were given by

$$
\lambda_r = \frac{\mu(r)r}{\varphi(r)L} \sum_{d \leq z \atop p|d \Rightarrow p \leq R \atop (d,r)=1} \frac{\mu(d)^2}{\varphi(d)}.
$$

Estimate $\lambda_r$ in the case that $z = R^t$ with $t > 1$.

3. Use Selberg’s sieve to prove that

$$
\pi(x; q, a) \leq \frac{(2 + o(1))x}{\varphi(q) \log(x/q)}.
$$

4. (a) For $\Im(\tau) > 0$, define $\theta(\tau) = \sum_{n \in \mathbb{Z}} e^{\pi in^2}\tau$. Show that $\theta(-1/\tau) = (-i\tau)^{1/2} \theta(\tau)$.

(b) Use the previous exercise and the Mellin transform to prove the functional equation for $\zeta$. 