1. Show that for every \( q \),
\[
\sum_{1 \leq a < q, (a,q)=1} e\left(\frac{a}{q}\right) = \mu(q).
\]

2. Use the large sieve to prove that
\[
\pi(x; q, a) \leq \frac{(2 + o(1)) x}{\varphi(q) \log(x/q)}
\]
where \( q < x \).

3. Let \( \chi \) be a non-trivial character modulo \( q \). Show that
\[
|L(1, \chi)| < 3 \log q.
\]

4. Show that for a prime \( p \),
\[
S(1, 1; p^2) = 2p \Re \left( \frac{2}{p^2} \right),
\]
where \( S(1, 1; p^2) \) denotes the Kloosterman sum
\[
S(1, 1; p^2) = \sum_{0 \leq x \leq p^2, (x,p^2)=1} e\left(\frac{x + \overline{x}}{p^2}\right).
\]

5. Let \( S \subset \{1, \ldots, N\} \) such that for all primes \( p \) the number of residue classes of \( S \) modulo \( p \) is at most \( \frac{2+1}{2} \); show that \( |S| \ll \sqrt{N} \) and compute the implied constant.

6. Let \( s \geq 1 \) be a constant and \( y = z^s \). Define
\[
\pi^+(y, z) := \sum_{n \leq y, p|n \Rightarrow p \geq z} 1 - \lambda(n)
\]
\[
\pi^-(y, z) := \sum_{n \leq y, p|n \Rightarrow p \geq z} 1 + \lambda(n),
\]
where \( \lambda(n) = (-1)^{\Omega(n)} \) and \( \Omega(n) \) denotes the number of prime factors of \( n \) counted with multiplicity.
(a) Show that

\[
\pi^+(y, z) = \pi^+(y, w) - \sum_{w \leq p < z} \pi^-(\frac{y}{p}, p),
\]

\[
\pi^-(y, z) = \pi^-(y, w) - \sum_{w \leq p < z} \pi^+(\frac{y}{p}, p).
\]

(b) Show that

\[
\pi^+(y, z) = F(s)\frac{y}{e^\gamma \log(z)} + 2H(s)\frac{y}{\log(z)^2} + O_s\left(\frac{y}{\log(z)^3}\right),
\]

\[
\pi^-(y, z) = f(s)\frac{y}{e^\gamma \log(z)} - 2h(s)\frac{y}{\log(z)^2} + O_s\left(\frac{y}{\log(z)^3}\right),
\]

where

\[
F(s) = \frac{2e^\gamma}{s}
\]

\[
\frac{d}{ds}(sF(s)) = f(s - 1)
\]

\[
f(s) = \frac{2e^\gamma \log(s - 1)}{s}
\]

\[
\frac{d}{ds}(sf(s)) = F(s - 1)
\]

\[
H(s) = \frac{1}{s^2}
\]

\[
\frac{d}{ds}(s^2H(s)) = -sh(s - 1)
\]

\[
h(s) = \frac{1}{s^2} \left( 1 + \frac{1}{s - 1} - \log(s - 1) \right)
\]

\[
\frac{d}{ds}(s^2h(s)) = -sH(s - 1)
\]

(c) Show that

\[
F(s) - 1 = O(s^{-s})
\]

\[
1 - f(s) = O(s^{-s}).
\]