1. Fill in the gaps in the proof of the Heath-Brown identity:

Let $J \geq 1$, $X \geq 1$, $1 \leq n \leq 2X$. Then

$$Λ(n) = -\sum_{j=1}^{J} (-1)^j \binom{J}{j} \sum_{m_1 \cdots m_j \leq X^{1/j}} \mu(m_1) \cdots \mu(m_j) \sum_{m_1 \cdots m_j n_1 \cdots n_j = n} \log n_1.$$ 

2. Show that for $A \in \mathbb{N}$, $\alpha \in \mathbb{R}/\mathbb{Z}$, and $x \geq 2$,

$$\sum_{n \leq x} \mu(n) e(n\alpha) \ll A \frac{x}{(\log x)^A}.$$ 

3. Consider the following random “stick breaking” process. Start with a stick of length 1, break off a piece of length $X_1$ uniformly distributed between 0 and 1, then break off a piece of the remaining stick of length $X_2$ uniformly distributed between 0 and $1 - X_1$, and so on: at the $n$th stage of the process we break off a piece of the remainder of the stick having length $X_n$ uniformly distributed between 0 and $1 - X_1 - \cdots - X_{n-1}$. Finally, we define the random set $S = \{X_1, X_2, \ldots\}$.

(a) Let $X$ be a random element of the set $S$, chosen according to the distribution $P[X = X_i] = X_i$. Show that $X$ is uniformly distributed in $[0, 1]$.

(b) Show that the probability that the largest element of $S$ is at most $1/u$ is equal to $\rho(u)$. Compute the expected size of the largest element of $S$.

(c) Let $N$ be large, and let $n$ be a random number between $N$ and $2N$. Write $n = p_1 \cdots p_k$ with each $p_i$ prime, and choose an index $1 \leq i \leq k$ at random, choosing $i$ with probability $\log(p_i) / \log(n)$. Let $X = \log(p_i) / \log(n)$. Show that as $N$ goes to infinity, the random variable $X$ becomes uniformly distributed in $[0, 1]$. Conclude that the the random set $\{\log(p_1) / \log(n), \ldots, \log(p_k) / \log(n)\}$ behaves like $S$ (in some sense) as $N$ goes to infinity.

(d) Let $n$ be large, and let $\sigma$ be a uniformly random permutation from $S_n$. Let $l$ be the length of the cycle of $\sigma$ which contains 1, and let $X = \frac{l}{n}$. Show that $P[l = i] = \frac{1}{n}$ for any integer $1 \leq i \leq n$. Conclude that when $n$ goes to infinity, $X$ becomes uniformly distributed in $[0, 1]$, and that the set of cycle lengths of $\sigma$ divided by $n$ behaves like $S$ (in some sense).

(e) Mimic part (c) to give an approximate description of the set of the degrees of the irreducible factors of a random polynomial of large degree in $\mathbb{F}_q[t]$.
4. Try to prove the following lemma of Zhang:

Let $n \geq 1$, $\frac{1}{10} < \sigma < \frac{1}{2}$, $t_1, \ldots, t_n$ with $\sum_i t_i = 1$. Then one of the following holds:

(i) One of the $t_i$ is $\geq \frac{1}{2} + \sigma$.
(ii) There exists a partition $S \cup T = \{1, \ldots, n\}$ with $S \cap T = \emptyset$ such that

\[
\frac{1}{2} - \sigma \leq \sum_{i \in S} t_i \leq \sum_{i \in T} t_i \leq \frac{1}{2} + \sigma.
\]

(iii) There exist $i, j, k$ distinct with

\[
2\sigma \leq t_i \leq t_j \leq t_k \leq \frac{1}{2} - \sigma
\]

and

\[
t_i + t_j, t_i + t_k, t_j + t_k \geq \frac{1}{2} + \sigma.
\]