Open problem: Is there an algorithm to decide whether a periodic sandpile in $\mathbb{Z}^2$ is explosive?

Lionel Levine

Let $s : \mathbb{Z}^2 \to \{0, 1, 2, 3\}$ be periodic, i.e.

$$s(x) = s(x + \lambda), \qquad x \in \mathbb{Z}^2, \lambda \in \Lambda$$

for a full rank sublattice $\Lambda \subset \mathbb{Z}^2$ (if you like, take $\Lambda = m\mathbb{Z} \times n\mathbb{Z}$).

We view s as a (stable) sandpile: each site $x \in \mathbb{Z}^2$ starts with $s(x)$ chips. Now fix a positive integer n, add n chips at the origin, and try to stabilize. This means: every time you see a site with 4 or more chips, you can topple it by sending 1 chip to each of the four nearest neighbors. Let $u(x)$ be the number of times x topples. (This number does not depend on the order of topplings.) Then either

1. $u(x) < \infty$ for all $x \in \mathbb{Z}^2$, and $u(x) = 0$ for some $x \in \mathbb{Z}^2$; or
2. $u(x) = \infty$ for all $x \in \mathbb{Z}^2$.

If case 2 holds for some $n < \infty$, then we say that s is *explosive*.

Problem. Find an algorithm that takes s as input and decides whether or not s is explosive.

Figure 1 shows a sandpile in the process of exploding. The figure is reproduced from [1], where you can find some more background and references. Here are a few things we know:

- If the *density*
$$\rho(s) := \frac{1}{|\mathbb{Z}^2/\Lambda|} \sum_{x \in \mathbb{Z}^2/\Lambda} s(x)$$
is strictly less than 2, then s is not explosive.

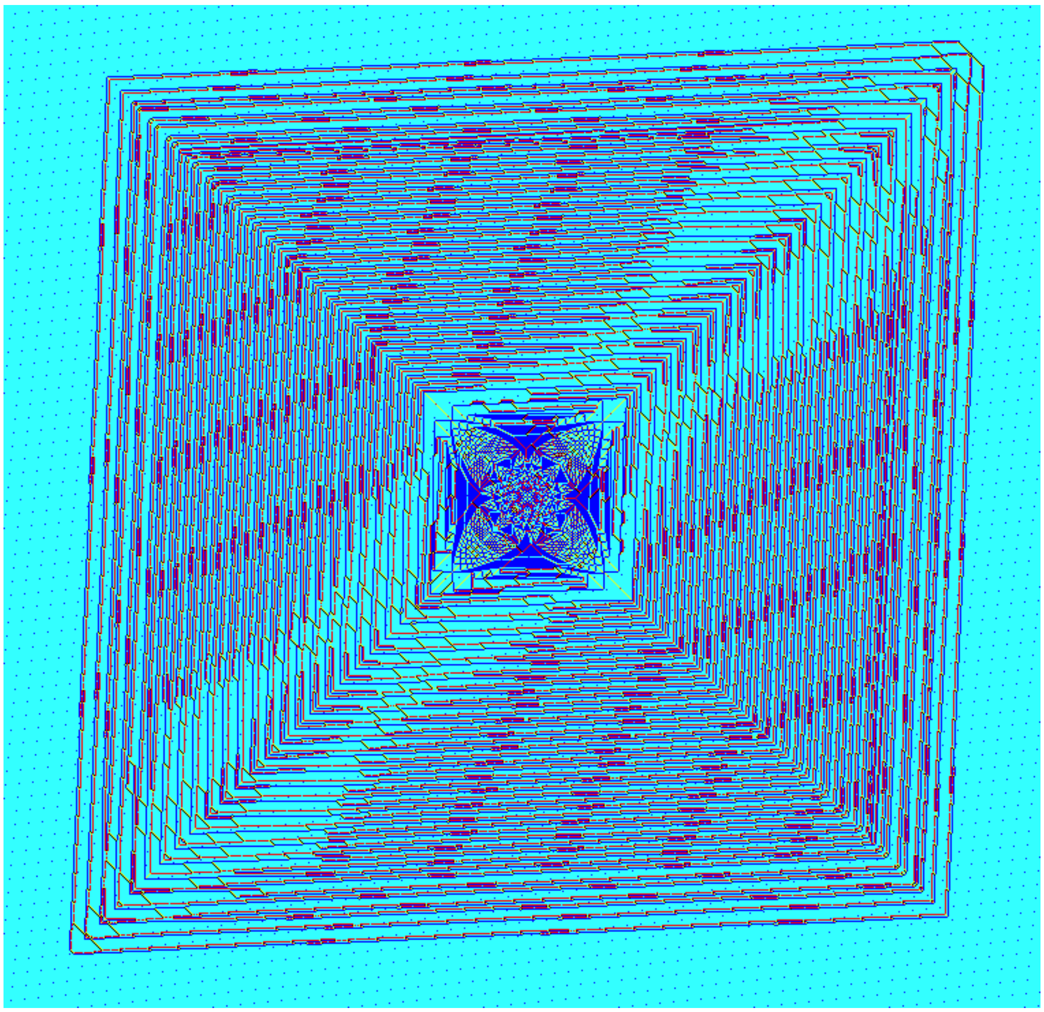

Figure 1: The sandpile $s = 2 + 1_\Lambda$ is explosive, where Λ is the lattice in $\mathbb{Z}^2$ spanned by $(1, 10)$ and $(10, 1)$. The image above was obtained by starting with s, adding $n = 5000$ chips at the origin and performing some topplings. Each pixel is colored according to the number of chips present at the corresponding site of $\mathbb{Z}^2$ (yellow 0, red 1, light blue 2, dark blue 3, black 4 or more). Every site in $\mathbb{Z}^2$ will topple infinitely often.

- If $s \leq 2$ on infinitely many rows and columns of $\mathbb{Z}^2$, then s is not explosive.
- If $s \geq 2$ everywhere and $\{s = 3\}$ meets every row and column of $\mathbb{Z}^2$, then s is explosive.

References

[1] Anne Fey, Lionel Levine and Yuval Peres, Growth rates and explosions in sandpiles, *J. Stat. Phys.* **138**:143–159, 2010. arXiv:0901.3805