2013 Summer Graduate Workshop, Cortona, Italy: Mathematical General Relativity
Some Scalar Curvature Basics

Recall that for better or worse, we defined the curvature tensor as

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

with $R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}) = R^\ell_{ijk}\frac{\partial}{\partial x^\ell}$.

Also, recall that a comma in a subscript denotes partial differentiation (with respect to some coordinate), whereas a semicolon in a subscript denotes covariant differentiation. For example, if h is a $(1,2)$-tensor with components in a coordinate chart $h^i_{\ jk}$, then the covariant derivative ∇h is a $(1,3)$-tensor with components

$$h^i_{\ jk;\ell} := \nabla h\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^\ell}\Big) = (\nabla_{\frac{\partial}{\partial x^\ell}} h)\Big(dx^i, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big) = h^i_{\ jk,\ell} + \Gamma^i_{m\ell} h^m_{\ jk} - \Gamma^m_{j\ell} h^i_{\ mk} - \Gamma^m_{k\ell} h^i_{\ jm}.$$

Problem 1. Linearization of Scalar Curvature. Let $R(g) = g^{ij}R_{ij}$ be the scalar curvature of a metric (not necessarily Riemannian). Consider a variation $g(t) = g + th$ of g in the direction of a symmetric $(0,2)$-tensor h (more generally, note that all you will use is that $g(t)$ is smooth in t, with $g(0) = g$ and $g'(0) = h$). For small t, $g(t)$ is a metric. Define $L_g(h) := DR_g(h) = \frac{d}{dt}\Big|_{t=0} R(g(t))$.

Derive the scalar curvature formula

$$R(g) = g^{ij}R_{ij} = g^{ij}\left(\Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik}\right)$$

and use it to verify the identity

$$L_g(h) = -\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g(\mathrm{div}_g(h)) - \langle h, \mathrm{Ric}(g)\rangle_g.$$

Hints: One approach is to compute in normal coordinates at a point p, so that $g_{ij}(p) = \delta_{ij}$, as well as $\Gamma^k_{ij}(p) = 0$, equivalently, $g_{ij,k}(p) = 0$. Indicated below are some formulas to guide you, and which *you should verify* as you compute. We emphasize here and below that all quantities are evaluated at p. In such normal coordinates, we have *at* p: $h_{ij;k\ell} = h_{ij,k\ell} - h_{mj}\Gamma^m_{ik,\ell} - h_{im}\Gamma^m_{jk,\ell}$. From this, one shows that *at* p,

$$-\Delta_g(\mathrm{tr}_g(h)) = -g^{k\ell}g^{ij}h_{ij;k\ell} = -\sum_{i,k}(h_{ii,kk} - 2h_{km}\Gamma^m_{ik,i})$$

$$\mathrm{div}_g(\mathrm{div}_g(h)) = g^{j\ell}g^{ik}h_{ij;k\ell} = \sum_{i,k}(h_{ik,ik} - h_{km}\Gamma^m_{ii,k} - h_{km}\Gamma^m_{ik,i}).$$

To find the variation of the scalar curvature, express the scalar curvature in terms of Christoffel symbols, and take a time derivative, expand and turn the crank. To find the derivative of the inverse metric, note that if $A(t)$ is a smooth curve in $GL(n)$, then $\frac{d}{dt}A^{-1}(t)$ is easily computed from $A(t)A^{-1}(t) = I$ using the product rule.

Another approach is to note that $\frac{d}{dt}\Big|_{t=0}\Gamma^k_{ij}$ are the components $(\delta\Gamma)^k_{ij}$ of a tensor $\delta\Gamma$ (this follows from #4 a on the earlier set on basic geometry problems), so that clearly the variation of the Ricci tensor is given by $\frac{d}{dt}\Big|_{t=0}R_{ij} = (\delta\Gamma)^k_{ij;k} - (\delta\Gamma)^k_{ik;j}$. One should now express $\delta\Gamma$ in terms of the covariant derivative of h.

Problem 2. Conformal deformation of scalar curvature.

a. Suppose (M^n, g) is a Riemannian metric and $\tilde{g} = e^{\varphi} g$. Show that

$$R(\tilde{g}) = e^{-\varphi}\left(R(g) - (n-1)\Delta_g \varphi - \frac{1}{4}(n-1)(n-2)|\nabla\varphi|_g^2\right).$$

b. In case $n \geq 3$, if we write $e^{\varphi} = u^{\frac{4}{n-2}}$ for $u > 0$, then

$$R(\tilde{g}) = u^{-\frac{n+2}{n-2}}\left(R(g)u - \frac{4(n-1)}{(n-2)}\Delta_g u\right).$$

c. Let $c(n) = \frac{n-2}{4(n-1)}$ and $L_g u = \Delta_g u - c(n)R(g)u$ is the *conformal Laplacian*, show that the total scalar curvature of $\tilde{g} = u^{\frac{4}{n-2}} g$ is given by

$$\int_M R(\tilde{g})\, dv_{\tilde{g}} = c(n)^{-1}\int_M \left(|\nabla u|_g^2 + c(n)R(g)u^2\right) dv_g.$$

Hint: Show that $dv_{\tilde{g}} = u^{\frac{2n}{n-2}}\, dv_g$.

Problem 3. Volume expansion of geodesic balls.

a. Suppose $(V, \langle\ ,\ \rangle)$ is an n-dimensional real inner product space. Suppose that $T : V \to V$ is a self-adjoint linear operator. If $d\sigma$ is the Euclidean area measure, $\mathbb{B}^n$ is the unit ball, and $\mathbb{S}^{n-1} \subset V$ is the unit sphere in V, then if vol is the Euclidean volume,

$$\int_{x\in\mathbb{S}^{n-1}} \langle T(x), x\rangle\, d\sigma = \operatorname{trace}(T)\ \operatorname{vol}(\mathbb{B}^n).$$

b. If (M, g) is Riemannian and $p \in M$, let $B_r(p) \subset M$ be the geodesic ball of radius $r > 0$ (for sufficiently small r). Then

$$\operatorname{vol}_g(B_r(p)) = \operatorname{vol}(\mathbb{B}^n) r^n\left[1 - \frac{R(g)|_p}{6(n+2)}\, r^2 + O(r^3)\right].$$

Hint: You may wish to observe and use $\det(I + tA) = 1 + t\ \operatorname{trace}(A) + O(t^2)$, along with Problem #4c below.

Problem 4. Metric expansion in normal coordinates. Suppose that $\gamma(t)$ is a unit-speed geodesic, and that $J(t)$ is a Jacobi field along γ: $J''(t) = R(\gamma'(t), J(t), \gamma'(t))$.

a. If R is the Riemann curvature tensor, show that

$$J'''(t) = (\nabla_{\gamma'(t)} R)(\gamma'(t), J(t), \gamma'(t)) + R(\gamma'(t), J'(t), \gamma'(t)).$$

b. Suppose $J(0) = 0$. Let $\chi(t) = \langle J(t), J(t)\rangle$. Derive the fourth-order Taylor expansion

$$\chi(t) = \sum_{k=0}^{4} \frac{\chi^{(k)}(0)}{k!} t^k + \mathcal{E}(t) = |J'(0)|^2 t^2 - \frac{1}{3}\langle R(\gamma'(0), J'(0), J'(0), \gamma'(0))\rangle t^4 + O(t^5).$$

c. Suppose (M, g) is a Riemannian manifold and $p \in M$. Show that in normal coordinates centered at p (so $x^i(p) = 0$)

$$g_{ij}(x) = \delta_{ij} - \frac{1}{3} R_{kij\ell} x^k x^\ell + O(|x|^3).$$

HINT: In normal coordinates (x^i) centered at p, consider a unit speed radial geodesic $\gamma(t)$ and the vector field $W(t) = tW^i \frac{\partial}{\partial x^i}$ along γ, where W^i are constants. Show that $W(t)$ is a Jacobi field along γ with $W'(0) = W^i \frac{\partial}{\partial x^i}\Big|_p$. One way to do this is to build a variation $\Gamma(s,t)$ of $\gamma = \Gamma(0, \cdot)$ through geodesics. In any case, you might first observe that in normal coordinates, the curve β with components $\beta^i(t) = tV^i$ is a geodesic with $\beta(0) = p$ and $\beta'(0) = V^i \frac{\partial}{\partial x^i}\Big|_p$.

PROBLEM 5. GEOMETRIC FORMULA FOR GAUSSIAN CURVATURE. Let (M, g) be a surface with a Riemannian metric g. Consider an orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$ of T_pM. Note that the Gauss curvature at p is just $K(p) = R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1)$, where R is the Riemann tensor. Consider a normal neighborhood of radius $a > 0$ about p, with normal coordinates (x, y) built off of the orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$ of T_pM: $(x, y) \mapsto \exp_p(x\mathbf{e}_1 + y\mathbf{e}_2)$. Define geodesic polar coordinates by $(r, \theta) \mapsto f(r, \theta) = \exp_p(r\cos\theta\ \mathbf{e}_1 + r\sin\theta\ \mathbf{e}_2)$. Note that the change of coordinates map is just $(r, \theta) \mapsto (x, y) = (r\cos\theta, r\sin\theta)$, which shows that the map f, which is clearly smooth for $r < a$ and all θ, is a diffeomorphism for $0 < r < a$ and $\theta \in I$, where I in any open interval of length at most 2π. Note that by the Gauss' Lemma, the metric components in geodesic polar coordinates are $g_{rr} = 1$, $g_{r\theta} = 0$ and $g_{\theta\theta} = \left|\frac{\partial f}{\partial \theta}\right|^2$. Since the radial curves of constant θ on M are geodesics, for any θ_0, $J(r) = \frac{\partial f}{\partial \theta}(r, \theta_0)$ is a Jacobi field along the radial geodesic $r \mapsto \gamma(r) = f(r, \theta_0)$.

a. For any θ_0, show that $J'(0) \perp \gamma'(0)$.

b. Use Problem 4 to show that $g_{\theta\theta}(r, \theta) = r^2 - \frac{K(p)}{3} r^4 + \mathcal{E}(r, \theta)$, where $\mathcal{E}(r, \theta) = O(r^5)$ uniformly in θ, i.e. $R(r, \theta) \le Cr^5$, C can be chosen independent of r and θ. Use the Taylor expansion $(1 + x)^\alpha = 1 + \alpha x + O(x^2)$ to derive

$$\sqrt{g_{\theta\theta}}(r, \theta) = r - \frac{K(p)}{3!} r^3 + O(r^4).$$

c. Let $L(r)$ be the length of a geodesic circle of radius r about p, and let $A(r)$ be the area enclosed by this circle, both computed using the metric g. Show that

$$\lim_{r \to 0^+} \frac{3}{\pi} \frac{2\pi r - L(r)}{r^3} = K(p) = \lim_{r \to 0^+} \frac{12}{\pi} \frac{\pi r^2 - A(r)}{r^4}.$$

d. Let D be the unit disk in the plane: $D = \{(x, y) : x^2 + y^2 < 1\}$. Consider the hyperbolic metric

$$g_H = \frac{4}{(1 - (x^2 + y^2))^2}(dx^2 + dy^2) = \frac{4}{(1 - \rho^2)^2}(d\rho^2 + \rho^2 d\theta^2)$$

where (ρ, θ) are standard polar coordinates on D. By solving the differential equation $\frac{2d\rho}{1-\rho^2} = dr$, show how to re-write the hyperbolic metric as $g_H = dr^2 + \sinh^2 r\ d\theta^2$. Use this along with the formulas above to show $K = -1$ at the origin of coordinates (of course, $K = -1$ everywhere). (Recall that $dr^2 = dr \otimes dr$.)