# Introduction to General Relativity and the Einstein Constraint Equations

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# Contents

*Chapter 1*

# A Brief Introduction to Special Relativity and Minkowski Space

## 1.1. The Lorentz Transformations

We begin by discussing some of the physical underpinnings of the Minkowski space model of physics as developed in the spirit of Einstein. From a mathematical point of view, we make the assumption that events in space and time form a four-dimensional continuum $M$, which for now we take to be $\mathbb{R}^4$ topologically. We discuss how Einstein identified preferred sets of coordinate charts which correspond to a certain class of *physical observers*. The transformations between such charts form a group of diffeomorphisms of $\mathbb{R}^4$, and from the Kleinian perspective, the invariants of this group yield *geometric* quantities of interest.

**1.1.1. Galilean transformations.** Inertial frames of reference (inertial observers) are those in which Newton's first law, *the law of inertia*, holds. The law states that objects will move with constant velocity unless acted upon by a force. This overturned the Aristotelian law, which stated that objects which are not acted upon by a force should naturally come to *rest*. Now, it is clear that an observer in uniform motion with respect to an inertial observer is also an inertial observer, and trajectories which correspond to constant-velocity paths move in straight lines in both space and space-time.

The principle of Galilean/Newtonian relativity is that one cannot detect absolute motion of any inertial reference frame, only relative motion:

physics should "look the same" to all inertial observers. In other words, all inertial systems are equivalent for the formulation of physical laws; there is no preferred inertial frame. As a reference frame is a coordinate chart for space-time, we look for those coordinate changes that correspond to comparison of measurements made by two inertial observers. It was assumed that the universe is endowed with some *Newtonian time function* that measures absolute *time intervals* between events. If we translate the times for two inertial observers, we can arrange that the time functions agree. We can write the transformation that relates the coordinates $(t, \mathbf{x})$ of an event (point in space-time) in one frame $\mathcal{O}$ to the coordinates $(\tilde{t}, \tilde{\mathbf{x}})$ in a frame $\tilde{\mathcal{O}}$ moving at constant velocity $\mathbf{v}$ with respect to $\mathcal{O}$:

$$\tilde{t} = t \tag{1.1}$$

$$\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{v}t. \tag{1.2}$$

If we also arrange the relative velocity to lie along the $x$-axis, then the transformation becomes, with $\mathbf{v} = v\frac{\partial}{\partial x}$,

$$\begin{aligned}\tilde{t} &= t\\ \tilde{x} &= x - vt\\ \tilde{y} &= y\\ \tilde{z} &= z.\end{aligned}$$

Note that the for two points $E_1$ and $E_2$ with respective coordinates $(t_1, \mathbf{x}_1)$ and $(t_2, \mathbf{x}_2)$ in $\mathcal{O}$, and correspondingly labeled coordinates in $\tilde{\mathcal{O}}$, the Euclidean spatial separation is preserved: $\|\mathbf{x}_1 - \mathbf{x}_2\| = \|\tilde{\mathbf{x}}_2 - \tilde{\mathbf{x}}_1\|$. Moreover, if $\widehat{\mathcal{O}}$ is an inertial observer moving with constant velocity $\mathbf{w}$ with respect to $\tilde{\mathcal{O}}$, then it is elementary to obtain the *Galilean law of addition of velocities*:

$$\mathbf{x} = \tilde{\mathbf{x}} - \mathbf{v}t = \hat{\mathbf{x}} - \mathbf{w}t - \mathbf{v}t = \hat{\mathbf{x}} - (\mathbf{v} + \mathbf{w})t. \tag{1.3}$$

We see that relative velocities satisfy a very simple addition rule.

Suppose one has a curve $\gamma : I \subset \mathbb{R} \to M$ parametrized by Newtonian time. If we let $\mathbf{x}(t)$ be the spatial components of $\gamma(t)$ in $\mathcal{O}$, and $\tilde{\mathbf{x}}(t)$ be the components in $\tilde{\mathcal{O}}$, then $\tilde{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{v}t$. Therefore, if a prime denotes a time derivative, $\tilde{\mathbf{x}}'(t) = \mathbf{x}'(t) - \mathbf{v}$, and $\tilde{\mathbf{x}}''(t) = \mathbf{x}''(t)$. Hence the acceleration $\mathbf{a}(t)$ of the path $\gamma$ is the same as measured in either frame. Moreover, if $\gamma$ is the path of an object of mass $m$ (which we take here to be independent of $t$), then $m\mathbf{a}(t)$ is the same in either frame. In many classical systems the force $\mathbf{F}$ between objects depends on the relative separation of the bodies, which is observer-independent, so that Newton's Second Law of Motion takes the same form in both frames.

Einstein's foundational 1905 paper is entitled *On the Electrodynamics of Moving Bodies*. Indeed the incompatibility of electromagnetism and the

Galilean transformations led to the reformulation of mechanics. There are of course very fundamental issues in interpreting electromagnetism. Nineteenth-century experiments revealed that a magnetic field is generated by charges in motion. The Lorentz force law $\mathbf{F} = q(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B})$ (written in *Gaussian* or cgs units (centimeters-grams-seconds)) determines the force on a charge $q$ moving at velocity $\mathbf{v}$ in an electromagnetic field. How do we interpret the physics if instead we switch to a frame *moving with* the charges generating the magnetic field, or to a frame in which the charge $q$ in the Lorentz law is stationary? Consider a circular wire loop which moving past a stationary magnet whose field lines move *through the loop*. The moving charges experience a force from the Lorentz force law. If instead we view the wire as stationary, but the magnet as moving, then the charges experience a force from as electric force induced (Faraday's Law) by the *changing* magnetic field. In the end, of course, the physical predictions are the same in each case, it is only the *interpretation* that differs. Einstein looked for a fundamental explanation of this in terms of relativity, that the laws of electromagnetism should have the same form in all inertial frames.

A consequence of Maxwell's equations for electromagnetism is that light travels according to a wave equation, the speed of which can be determined. Of course, this begs the question: the speed relative to what? And what was the medium capable of transmitting electromagnetic disturbances at such a great speed, while seeming transparent to the motion of the earth through it? Experiments such as the Michelson-Morley experiment in the late nineteenth century failed to find the medium, a *preferred* reference frame (which was called the *ether frame*), with respect to which light in vacuum travels at, well, the speed of light—roughly $3 \times 10^8$ meters per second. Under the Galilean transformations, inertial observers in relative motion with respect to the ether frame would have different measurements of the value of the speed of light. That this was not observed in experiments caused quite a quandary. Other experiments demonstrated that hypotheses which were proposed to explain the result of Michelson-Morley did not hold, such as the Kennedy-Thorndike experiment for the Lorentz contraction hypothesis, and the classical results on stellar aberration and the Fizeau experiment, which supplied evidence against a theory that the earth moved along with the ether. The ether theory embraced the notion that the principle of relativity did not apply to electromagnetism, in the sense that the ether frame is a preferred frame of reference for the theory. That there were problems with this led Einstein to postulate that relativity does apply to electromagnetism. Thus, although Newtonian dynamics works well with the Galilean transformations, for relativity to apply to electromagnetism, the Galilean transformations required modification.

For some foreshadowing, consider a function $\psi : M \to R$ which satisfies the wave equation (with wave speed $c$) in $\mathcal{O}$, in the sense that

$$\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}.$$

Let's suppose the relative velocity is aligned along the $x$-axis. Then since

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial\tilde{t}}{\partial t}\frac{\partial}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial t}\frac{\partial}{\partial\tilde{x}} = \frac{\partial}{\partial\tilde{t}} - v\frac{\partial}{\partial\tilde{x}} \\ \frac{\partial}{\partial x} &= \frac{\partial\tilde{t}}{\partial x}\frac{\partial}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}} = \frac{\partial}{\partial\tilde{x}}, \end{aligned}$$

we see that $\Psi(\tilde{t}, \tilde{\mathbf{x}}) := \psi(t, \mathbf{x})$ satisfies

$$\frac{1}{c^2}\left(\frac{\partial}{\partial\tilde{t}} - v\frac{\partial}{\partial\tilde{x}}\right)^2(\Psi) = \frac{\partial^2\Psi}{\partial\tilde{x}^2} + \frac{\partial^2\Psi}{\partial\tilde{y}^2} + \frac{\partial^2\Psi}{\partial\tilde{z}^2}.$$

This can be re-written as

$$\frac{1}{c^2}\frac{\partial^2\Psi}{\partial\tilde{t}^2} - 2\frac{v}{c^2}\frac{\partial^2\Psi}{\partial\tilde{t}\partial\tilde{x}} = \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2\Psi}{\partial\tilde{x}^2} + \frac{\partial^2\Psi}{\partial\tilde{y}^2} + \frac{\partial^2\Psi}{\partial\tilde{z}^2}.$$

We see that the wave equation is not invariant under Galilean transformations. This is perfectly reasonable for mechanical waves, but the homogeneous wave equation governing light propagation in vacuum is a consequence of Maxwell's equations, and we have seen that experiments indicate that light travels at the same speed in vacuum for all inertial observers.

However, it is not too hard to play around with the transformation so as to coax the preceding equation into the standard wave equation form. Namely, let

$$\begin{aligned} \tilde{t} &= \frac{1}{\sqrt{1-(v/c)^2}}(t - \frac{v}{c^2}x) \\ \tilde{x} &= \frac{1}{\sqrt{1-(v/c)^2}}(x - vt) \end{aligned}$$

replace the Galilean coordinate change. Then as we did above, we obtain

$$\begin{aligned} \frac{\partial}{\partial t} &= \frac{\partial\tilde{t}}{\partial t}\frac{\partial}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial t}\frac{\partial}{\partial\tilde{x}} = \frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{\partial}{\partial\tilde{t}} - v\frac{\partial}{\partial\tilde{x}}\right) \\ \frac{\partial}{\partial x} &= \frac{\partial\tilde{t}}{\partial x}\frac{\partial}{\partial\tilde{t}} + \frac{\partial\tilde{x}}{\partial x}\frac{\partial}{\partial\tilde{x}} = \frac{1}{\sqrt{1-(v/c)^2}}\left(\frac{\partial}{\partial\tilde{x}} - \frac{v}{c^2}\frac{\partial}{\partial\tilde{t}}\right). \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{c^2}\frac{\partial^2}{\partial t^2} &= \frac{1}{c^2}\cdot\frac{1}{(1-(v/c)^2)}\left(\frac{\partial^2}{\partial\tilde{t}^2} - 2v\frac{\partial^2}{\partial\tilde{t}\partial\tilde{x}} + v^2\frac{\partial^2}{\partial\tilde{x}^2}\right) \\ \frac{\partial^2}{\partial x^2} &= \frac{1}{(1-(v/c)^2)}\left(\frac{\partial^2}{\partial\tilde{x}^2} - 2\frac{v}{c^2}\frac{\partial^2}{\partial\tilde{t}\partial\tilde{x}} + \frac{1}{c^2}\left(\frac{v}{c}\right)^2\frac{\partial^2}{\partial\tilde{x}^2}\right). \end{aligned}$$

Therefore,

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2}{\partial \tilde{t}^2} - \frac{\partial^2}{\partial \tilde{x}^2}.$$

We found a coordinate change for which the wave equation is preserved, but at what cost? Especially disturbing is that $\tilde{t}$ depends not only on $t$, but on $x$ and $v$ as well! In fact, we will derive these equations from applying a few simple fundamental principles, as Einstein did. Namely, we combine the Galilean/Newtonian principle that physics should have the same form in all inertial frames along with Einstein's postulate that the speed of light in a vacuum is a physical law, and thus the same for all inertial observers. This immediately mitigates the need for an ether, and is consistent with the results of Michelson and Morley.

**1.1.2. Deriving the Lorentz Transformations.** We will now derive the Lorentz transformation of coordinates between inertial observers, as well as the transformation for electromagnetic fields, which allows one to deduce that Maxwell's equations constitute a physical law as well. We suppose that the origins in each coordinate system correspond to the same point in $M$. Straight line paths in one coordinate system represent uniform motion, and should therefore be straight lines in the other coordinate system (by the Law of Inertia and the Principle of Relativity). Since the origins match up, and if we assume the coordinate transformation is continuous, then we can conclude with a little work that the transformation must be linear. Another way to argue for linearity is based on the principle that space-time should be isotropic, i.e. there should be no preferred directions or points. A nonlinear change of coordinates would not preserve displacements along some line parallel to an axis (space or time), which would violate isotropy. Other points of view are to postulate linearity either for simplicity's sake, or in the sense of a Taylor approximation, and see what happens. As we are about to see, linear maps exist which do the job!

The paths of light rays emanating from the origin comprise the *light cone*, which can be represented in coordinates as $\Sigma := \{(t, \mathbf{x}) : (ct)^2 = \|\mathbf{x}\|^2\}$. Applying the Einstein postulate that the speed of light in vacuum is the same for all inertial observers, we require that this linear transformation *preserve the light cones*. It is reasonable to propose, based on the assumption that there is no preferred direction orthogonal to the direction of motion, that in directions orthogonal to the direction of motion the coordinates are the same for both observers, i.e. we can set up axes so that the direction of relative motion is along the $x$-axes in each coordinate system, and moreover $\tilde{y} = y$ and $\tilde{z} = z$. Indeed, you don't expect to lose the orthogonality of spatial vectors, since if the angles change one way in a moving frame, then they ought to change the same way if we *reverse* the direction of motion. But if

$T_{\mathbf{v}}$ represents the linear map we are looking for, the Principle of Relativity suggests $T_{\mathbf{v}}^{-1} = T_{-\mathbf{v}}$. Actually one might more generally argue that the transformation in these directions could be a dilation, but this would also not be compatible the Principle of Relativity. Indeed, consider two points that differ by a spatial displacement of unit size in one frame of reference $\mathcal{O}$. If an observer $\tilde{\mathcal{O}}$ moving orthogonally to this displacement would measure the spatial displacement of the points to be different than one, let's say less than one, for example, then by now reversing the roles, we should have that $\mathcal{O}$, which is moving with respect to $\tilde{\mathcal{O}}$, should measure the displacement to be less than that measured by $\tilde{\mathcal{O}}$—obviously this cannot be the case.

We can in fact then reduce the problem to a two-dimensional problem, since by considerations as above, a vector along the light cone in the $(t, x)$-plane should map to the $(\tilde{t}, \tilde{x})$ plane. We let the reduced mapping be $T_v$. Using standard bases we represent the linear transformation by a matrix $[T_v] = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$, so that

$$\begin{aligned} t &= \alpha\tilde{t} + \beta\tilde{x} \\ x &= \gamma\tilde{t} + \delta\tilde{x}. \end{aligned}$$

Now $\tilde{x} = 0$ is the path of the observer moving with respect to $\mathcal{O}$ with velocity $v$, and is given by $t = \alpha\tilde{t}$, $x = \gamma\tilde{t}$, so that $v = \gamma/\alpha$. Now we apply the invariance of the light cone, which implies that the vectors $\begin{bmatrix} 1 \\ c \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -c \end{bmatrix}$ are eigenvectors of $T_v$. In fact since the path $x = ct$ should map to $\tilde{x} = \tilde{c}t$ (positive directions correspond by orientation), the eigenvalues should be positive. This implies there are respective eigenvalues $\lambda_\pm > 0$ so that

$$\begin{aligned} \alpha \pm c\beta &= \lambda_\pm \\ \gamma \pm c\delta &= \pm c\lambda_\pm. \end{aligned}$$

Thus $c\alpha + c^2\beta = \gamma + c\delta$, and $c\alpha - c^2\beta = -\gamma + c\delta$. Solving these two equations yields $\alpha = \delta$, $\gamma = c^2\beta$. Using $v = \gamma/\alpha$ we then see the matrix has the form

$$[T_v] = \begin{bmatrix} \alpha & \beta \\ c^2\beta & \alpha \end{bmatrix} = \alpha(v) \begin{bmatrix} 1 & \frac{v}{c^2} \\ v & 1 \end{bmatrix}.$$

Now $0 < \lambda_+\lambda_- = \det[T_v] = (\alpha(v))^2(1 - (v/c)^2)$, so that we have $|v| < c$—note that the relative speed of two inertial observers is less than that of light. We also see $\alpha(v) > 0$, because $2\alpha(v) = \operatorname{tr}[T_v] = \lambda_+ + \lambda_- > 0$.

Consider the parity operator $P(t, x) = (t, -x)$. Suppose one applies the parity operation to a frame of reference, then considers this frame moving with velocity $-v$ with respect to a second frame. If one now applied the parity operator to the second frame, one sees that this situation is physically equivalent to the original situation of moving the first frame with velocity $v$ relative to the second frame. Thus we can see that the map $P \circ T_{-v} \circ P$ must be equal to $T_v$. Since $\det P = -1$, $\det T_{-v} = \det T_v$, But since $T_v \circ T_{-v} = I$, we see $\det T_v = \pm 1$, and thus $\det T_v = 1$ since we know the determinant is positive.

Together with the result of the last paragraph we see $\alpha(v) = \frac{1}{\sqrt{1-(v/c)^2}}$, and we note then that $\alpha(v) = \alpha(-v)$. Thus we have

$$[T_v] = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 & \frac{v}{c^2} \\ v & 1 \end{bmatrix}, \qquad [T_{-v}] = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 & -\frac{v}{c^2} \\ -v & 1 \end{bmatrix}.$$

Hence we have arrived at the Lorentz transformation

$$t = \frac{1}{\sqrt{1-(v/c)^2}} \left(\tilde{t} + \frac{v}{c^2}\,\tilde{x}\right), \qquad x = \frac{1}{\sqrt{1-(v/c)^2}} \left(v\tilde{t} + \tilde{x}\right) \tag{1.4}$$

$$\tilde{t} = \frac{1}{\sqrt{1-(v/c)^2}} \left(t - \frac{v}{c^2}\,x\right), \qquad \tilde{x} = \frac{1}{\sqrt{1-(v/c)^2}} \left(-vt + x\right). \tag{1.5}$$

Note that if we change the first variable to $x^0 := ct$, so that the coordinates have the same units, we have

$$x^0 = \frac{1}{\sqrt{1-(v/c)^2}} \left(\tilde{x}^0 + \frac{v}{c}\,\tilde{x}\right), \qquad x = \frac{1}{\sqrt{1-(v/c)^2}} \left(\frac{v}{c}\tilde{x}^0 + \tilde{x}\right) \tag{1.6}$$

$$\tilde{x}^0 = \frac{1}{\sqrt{1-(v/c)^2}} \left(x^0 - \frac{v}{c}\,x\right), \qquad \tilde{x} = \frac{1}{\sqrt{1-(v/c)^2}} \left(-\frac{v}{c}x^0 + x\right). \tag{1.7}$$

Let $\beta = v/c$ (where $\beta$ is not to be confused with the "$\beta$" used in the derivation above), so that the matrix for $T_v$ relative to the bases $\left\{c\frac{\partial}{\partial \tilde{t}}, \frac{\partial}{\partial \tilde{x}}\right\}$ and $\{c\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\}$ is then just $[T_v] = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix}$, and likewise $[T_{-v}] = \frac{1}{\sqrt{1-\beta^2}} \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}$. For $-1 < \beta = v/c < 1$, there is a unique $\theta$ so that $\sinh\theta = \frac{\beta}{\sqrt{1-\beta^2}} = \frac{v/c}{\sqrt{1-(v/c)^2}}$. Then $\cosh\theta = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-(v/c)^2}}$ (since $\cosh\theta > 0$ and $\cosh^2\theta - \sinh^2\theta = 1$). Thus we can write from the above $[T_{-v}] = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$, so that if $\theta$ corresponds to $v$, $-\theta$ corresponds to $-v$.

We can use this to find the velocity addition rule in special relativity: if $\theta_1$ and $\theta_2$ correspond to $v_1$ and $v_2$, then we note that if we consider frame $\tilde{\mathcal{O}}$ moving along the $x$-axis of $\mathcal{O}$ at velocity $v_1$, and $\hat{\mathcal{O}}$ moving along the $\tilde{x}$ axis of $\tilde{\mathcal{O}}$ at velocity $v_2$, then $\hat{\mathcal{O}}$ is moving along the $x$-axis relative to $\mathcal{O}$ at velocity $v$ which satisfies the following:

$$\begin{aligned}\begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix} &= [T_{-v}] = [T_{-v_1}][T_{-v_2}] \\ &= \begin{bmatrix} \cosh\theta_1 & -\sinh\theta_1 \\ -\sinh\theta_1 & \cosh\theta_1 \end{bmatrix}\begin{bmatrix} \cosh\theta_2 & -\sinh\theta_2 \\ -\sinh\theta_2 & \cosh\theta_2 \end{bmatrix} \\ &= \begin{bmatrix} \cosh(\theta_1+\theta_2) & -\sinh(\theta_1+\theta_2) \\ -\sinh(\theta_1+\theta_2) & \cosh(\theta_1+\theta_2) \end{bmatrix}\end{aligned}$$

Thus we see that the set of Lorentz transformations corresponding to inertial observers in relative motion along a common axis form a group. Moreover, we also have from the above,

$$\frac{v}{c} = \tanh\theta = \tanh(\theta_1+\theta_2) = \frac{\tanh\theta_1 + \tanh\theta_2}{1+\tanh\theta_1\tanh\theta_2} = \frac{\frac{v_1}{c}+\frac{v_2}{c}}{1+\frac{v_1 v_2}{c^2}}.$$

Thus the Galilean rule for the addition of velocities, $v = v_1 + v_2$, is replaced by $v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$. It is easy to show that for $|v_1|, |v_2| < c$, it follows that $\left(1 \pm \frac{v_1}{c}\right)\left(1 \pm \frac{v_2}{c}\right) > 0$, so that again $|v| < c$.

## 1.2. Minkowski space

**1.2.1. The Minkowski metric.** We note that if we consider the matrix $\Lambda = \begin{bmatrix} \cosh\theta & -\sinh\theta \\ -\sinh\theta & \cosh\theta \end{bmatrix}$, then it follows easily that

$$\Lambda^T \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \Lambda = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

Thus we see that each element of the group of Lorentz transformations preserves the bilinear form $-(dx^0)^2 + dx^2 = -c^2 dt^2 + dx^2$. Thus from the Kleinian perspective, the Lorentz transformations give rise to a geometry, and the geometric quantities are those objects, such as the above bilinear form, which are preserved by the group of transformations.

If we return to higher dimensions, such as three spatial dimensions, then the relevant bilinear form is

$$\eta = -(dx^0)^2 + dx^2 + dy^2 + dz^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2.$$

The relevant group of Lorentz transformations is given by the connected component of the identity in the group of $\eta$-preserving linear transformations. We also obtain the Poincaré group of $\eta$-preserving affine transformations as a semi-direct product of the group of Lorentz transformations with the set of space-time translations.

We note that we will often let $\langle \mathbf{v}, \mathbf{w} \rangle := \eta(\mathbf{v}, \mathbf{w})$.

**1.2.2. Causal nature of vectors.** Vectors $\mathbf{w} = (c\Delta t, \Delta \mathbf{x})$ with $\eta(\mathbf{w}, \mathbf{w}) < 0$ are called *time-like*, since $\|\Delta \mathbf{x}\| < c|\Delta t|$. Then $|v| = \frac{\|\Delta \mathbf{x}\|}{|\Delta t|} < c$. Such vectors may thus represent the tangents to space-time paths of material particles. We note in (1.8) below how such vectors represent the space-time displacements between pairs of events that in some inertial coordinates share the same spatial coordinates. Similarly, *null vectors* $\mathbf{w}$ satisfy $\eta(\mathbf{w}, \mathbf{w}) = 0$; these are tangent to the light-cone and represent paths of light rays. Finally, vectors with $\eta(\mathbf{w}, \mathbf{w}) > 0$ are called *space-like*, and represent displacements between pairs of events which are simultaneous (i.e. share the same time coordinate) in some inertial coordinate system, see (1.9) below.

Suppose that $\mathbf{w} = (c\Delta t, \Delta x, 0, 0)$ is time-like, and let $v = \frac{\Delta x}{\Delta t}$, so that $|v| < c$. The Lorentz transformation $T_{-v}$ satisfies

$$(1.8) \qquad [T_{-v}] \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix} = \begin{bmatrix} * \\ 0 \end{bmatrix}.$$

Similarly if $|\Delta x| > c|\Delta t|$, then let $\frac{v}{c} = \frac{c\Delta t}{\Delta x}$, so that $|v| < c$. Then

$$(1.9) \qquad [T_{-v}] \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix} = \frac{1}{\sqrt{1-(v/c)^2}} \begin{bmatrix} 1 & -\frac{v}{c} \\ -\frac{v}{c} & 1 \end{bmatrix} \begin{bmatrix} c\Delta t \\ \Delta x \end{bmatrix} = \begin{bmatrix} 0 \\ * \end{bmatrix}.$$

1.2.2.1. *Twin paradox.* It turns out that the familiar triangle inequality for vectors in Euclidean geometry is *reversed* for time-like vectors in Minkowski space. Given a smooth time-like curve $\gamma(\lambda)$, we define the proper time $\Delta\tau$ along a portion of $\gamma$ as

$$\Delta\tau = c^{-1} \int_{\lambda_0}^{\lambda_1} \sqrt{-\langle \gamma'(\lambda), \gamma'(\lambda) \rangle}\, d\lambda,$$

and the proper time function as

$$\tau(\lambda) = c^{-1} \int_{\lambda_0}^{\lambda} \sqrt{-\langle \gamma'(\widehat{\lambda}), \gamma'(\widehat{\lambda}) \rangle}\, d\widehat{\lambda}.$$

If we re-parametrize $\gamma$ by proper time $\tau$ by inverting to get $\lambda = \lambda(\tau)$, so that $\tilde{\gamma}(\tau) := \gamma(\lambda(\tau))$, we have

$$\tilde{\gamma}'(\tau) = \gamma'(\lambda(\tau)) \frac{d\lambda}{d\tau} = \frac{c}{\sqrt{-\langle \gamma'(\lambda(\tau)), \gamma'(\lambda(\tau)) \rangle}} \gamma'(\lambda(\tau)).$$

Thus we see the tangent vector has constant length $\sqrt{-\langle \tilde{\gamma}'(\tau), \tilde{\gamma}'(\tau) \rangle} = c$. This means that the parameter $\tau - \tau(\lambda_0)$ is indeed the proper time elapsed along $\gamma$ from $\lambda_0$ to $\lambda(\tau)$.

We begin with the reversed triangle inequality. If $\overrightarrow{OB}$ is time-like, let $|\overrightarrow{OB}| = \sqrt{-\langle\overrightarrow{OB},\overrightarrow{OB}\rangle}$.

**Definition 1.1.** A vector is *causal* if it is time-like or null. A path is *causal* if at each point, its tangent vector is a time-like or null vector.

**Proposition 1.2.** *If $\overrightarrow{OB}$ future-pointing and time-like, and $\overrightarrow{OA}$ and $\overrightarrow{AB}$ are future-pointing and causal, then $|\overrightarrow{OB}| \geq |\overrightarrow{OA}| + |\overrightarrow{AB}|$, with equality only in case $O$, $A$ and $B$ are collinear.*

**Proof.** By applying a Lorentz transformation as in (1.8), we may assume as above that $B$ has coordinates $(t_B, 0, 0, 0)$, for $t_B > 0$, and that $A$ has coordinates $(t_A, x_A, 0, 0)$, with $|x_A| \leq ct_A$. Then $\overrightarrow{AB}$ has components $\langle t_B - t_A, -x_A, 0, 0\rangle$, with $|x_A| \leq c(t_B - t_A)$. Then $|\overrightarrow{OA}|^2 = (ct_A)^2 - x_A^2$, and $|\overrightarrow{AB}|^2 = c(t_B - t_A)^2 - x_A^2$. Then

$$\begin{aligned} |\overrightarrow{OA}| + |\overrightarrow{AB}| &= \sqrt{(ct_A)^2 - x_A^2} + \sqrt{c(t_B - t_A)^2 - x_A^2} \\ &\leq ct_A + c(t_B - t_A) \\ &= ct_B = |\overrightarrow{OB}|. \end{aligned}$$

The only way equality holds is if $x_A = 0$. □

The reversed triangle inequality has the following analogue for piece-wise smooth paths.

**Proposition 1.3.** *Suppose $O$ and $B$ are two points in Minkowski space so that the displacement vector $\overrightarrow{OB}$ is future-pointing and time-like. Then amongst all piecewise smooth future-pointing causal paths from $O$ to $B$, the one of maximal proper time interval is the straight-line path from $O$ to $B$.*

**Proof.** By applying a Lorentz transformation as in (1.8), we may choose inertial coordinates that $O$ is the origin and $B$ lies on the positive $t$-axis. If $\gamma$ is a time-like curve from $O$ to $B$ parametrized by proper time $\tau$, with coordinates $(t(\tau), x(\tau), y(\tau), z(\tau))$, then along $\gamma$,

$$\begin{aligned}\Delta\tau &= c^{-1}\int_0^{\Delta\tau} \sqrt{-\langle\gamma'(\tau),\gamma'(\tau)\rangle}\, d\tau \\ &= c^{-1}\int_0^{\Delta\tau} \sqrt{c^2\big(\frac{dt}{d\tau}\big)^2 - \big(\frac{dx}{d\tau}\big)^2 - \big(\frac{dy}{d\tau}\big)^2 - \big(\frac{dz}{d\tau}\big)^2}\, d\tau \\ &\leq \int_0^{\Delta\tau} \frac{dt}{d\tau} d\tau = \Delta t.\end{aligned}$$

We used the fact that $\frac{dt}{d\tau} > 0$. Also, if the curve were piecewise smooth, we could break it up into finitely many intervals and apply the above analysis on each interval. The inequality is clearly strict unless $x$, $y$ and $z$ are constant (equal to 0), so that the curve is along the straight line path. □

The title of the subsection refers to the following physical interpretation of the reversed triangle inequality. Suppose two twins are together at $O$, and both are at that time inertial observers moving relative to each other along their $x$-axes at velocity 80% of the speed of light. Suppose that from the point of view of one of the twins who maintains an inertial frame $\mathcal{O}$, the other twin travels for five years to arrive at space-time point $A$, quickly turns around and returns to join the other twin at space-time point $B$ after a total travel time of ten years as determined in $\mathcal{O}$. In other words, the proper time $|\overrightarrow{OB}|$ elapsed from $O$ to $B$ is ten years. On the other hand, the proper time that elapses on the other twin's path from $O$ to $A$ to $B$ is strictly less than ten years, by the previous corollary: the other twin is *younger*! How can this be if both are moving relative to each other, aren't the situations symmetric? Well, no: physically, the twin moving to point $A$ "and back" must accelerate to turn around. This acceleration means that this twin does not remain in an inertial frame the entire time. In other words, the two frames of reference (coordinate charts for Minkowksi space) do not overlap as a simple Lorentz transformation.

Before we move on, let's compute how much time passes for the other twin. Suppose the twins mark each passing year by sending a light signal to each other. From the point of view of $\mathcal{O}$, a signal sent at year $t = k$ will be received by the other twin on the *outgoing* part of the journey at time $t_k$ determined by $vt_k = c(t_k - k)$, which determines how long it will take the light to catch up to the moving twin. Now, $\frac{v}{c} = 0.8$, so $t_k = 5k$. Thus we see that the first signal ($k = 1$) will be received just as the second twin turns around to begin the return trip. Thus only one signal from $\mathcal{O}$ will be received on the outgoing portion of the trip, and so the other nine signals will be received by the other twin on the return portion of the journey. What about signals sent from the twin moving relative to $\mathcal{O}$? The first signal is sent after one unit of proper time has passed along the path from $O$ to $A$, which is at time $t = \frac{1}{\sqrt{1-(v/c)^2}}$ as measure in $\mathcal{O}$. The distance between the twins at this instant, as measured in $\mathcal{O}$ is just $\frac{v}{\sqrt{1-(v/c)^2}}$. Thus the time at which the signal will arrive at $x = 0$ is

$$\frac{1}{\sqrt{1-(v/c)^2}} + \frac{1}{c}\frac{v}{\sqrt{1-(v/c)^2}} = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}.$$

Similarly the signal sent after $\ell$ years of proper time have elapsed on the outward journey from $O$ to $A$ will arrive at $x = 0$ at $t = \frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}\ell$. Note that the time interval between the reception of successive signals is $\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}}$, which thus marks the relative frequency between the emission of signals (as measured by the emitter), and the reception of signals (as measured by the receiver). This difference in relative frequency is known as the *Doppler shift*. Now for $\frac{v}{c} = 0.8$, $\frac{\sqrt{1+v/c}}{\sqrt{1-v/c}} = 3$. The numerology works out so that the third signal sent along the path from $O$ to $A$ occurs at $t = \frac{3}{\sqrt{1-(0.8)^2}} = 5$, so three signals are sent along the outward journey, and the third signal arrives at $x = 0$ at $t = 9$. The twin at $A$ has sent three signals back to his twin, but has received only one signal. On the "homeward" journey, the twin will send three more signals (the last just as the twins are back together again), and we receive a total of nine signals from the other twin.

**1.2.3. Simultaneity.** The world-lines of inertial observers are special paths, namely time-like *geodesics*. Moreover, inertial observers observers correspond to certain coordinate charts on Minkowski space-time $\mathbb{M}$. Mathematically it is not a big deal a points in $\mathbb{M}$ has two different sets of coordinates in two different charts. However, interpreting the coordinates in the physical model yields some interesting results—the coordinates are not merely labels, but rather they are supposed to be the results of physical measurements.

The first observation is that *simultaneity is relative*: two different observers $\mathcal{O}$ and $\tilde{\mathcal{O}}$ moving relative to each other will not agree in general on whether two events occur at the same time. Imagine we synchronize the observers at the origin at an initial time, and that they are moving along their respective $x$-axes. The Lorentz transformations tell us that the events that $\tilde{\mathcal{O}}$ chart as occurring simultaneously at $\tilde{t} = 0$ correspond to $t = \frac{v}{c^2}x$ in $\mathcal{O}$. Different points in this set have different $t$-coordinates, and so $\mathcal{O}$ will not agree that they occur at the same time.

Though simultaneity is relative, both observers will calculate the same value of $-(ct)^2 + x^2 = -(x^0)^2 + x^2 = -(\tilde{x}^0)^2 + (\tilde{x})^2 = -(c\tilde{t})^2 + (\tilde{x})^2$. This observation can be used to obtain two interesting conclusions that can be verified experimentally: time dilation and Lorentz contraction.

Consider the event $A$ which has coordinates $\tilde{t} = 1$, $\tilde{x} = 0$ (we suppress the other spatial dimensions, whose coordinates we take to be 0). The Lorentz transformations give us the coordinates for $A$ in $\mathcal{O}$, and in particular $t = \frac{\tilde{t}+\frac{v}{c^2}\tilde{x}}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-(v/c)^2}} > 1$. $\mathcal{O}$ measures more time to have elapsed, and so concludes that the moving clock in $\tilde{\mathcal{O}}$'s frame *runs slow.* Another way to see this from the invariant hyperbola is to note that since at $A$, $-(c\tilde{t})^2 + \tilde{x}^2 = -c^2$, but $x = vt \neq 0$, then for $(-ct)^2 + x^2 = -c^2$ at $A$, $t > \tilde{t}$! You can see this with a simple picture: draw the hyperbola through $A$, and it intersects the $t$ axis at a coordinate *lower* than $t(A)$. In general, if time $\Delta\tilde{t}$ is measured between events at a fixed $\tilde{x}$ value, then the time between the events as measure in $\mathcal{O}$ will be $\Delta t = \frac{\Delta\tilde{t}}{\sqrt{1-(v/c)^2}} > \Delta\tilde{t}$.

Similarly, moving objects contract along the direction of motion. To be precise, consider a rod along the $\tilde{x}$-axis, whose rest length measured in $\tilde{\mathcal{O}}$ is $L$, and which is moving with velocity $v > 0$ along the $x$-axis. This means that the ends of the rod are measured simultaneously in $\tilde{\mathcal{O}}$ at, say, $O$ given by $(\tilde{t}, \tilde{x}) = (0, 0)$ and $A$ given by $(\tilde{t}, \tilde{x}) = (0, L)$. The ends of the rod make paths in space-time, one given by $\tilde{x} = 0$, the other by $\tilde{x} = L$. We need to find the coordinates of the point $B$ where $\tilde{x} = L$ intersects $t = 0$, since then both $O$ and $B$ will be simultaneous with respect to $\mathcal{O}$. By the Lorentz transformations, the event $B$ will have $\tilde{t} = -\frac{v}{c^2}L$, so that $x = \sqrt{1-(v/c)^2}\, L < L$. In $\mathcal{O}$, the rod is measured to have length $\sqrt{1-(v/c)^2}\, L$, since determining the length of the rod amounts to finding the spatial separation between the ends *at a fixed time*. It is this last issue that is relative. Note also that this length contraction can also be seen geometrically in terms of the invariant hyperbolae. Indeed, the hyperbola $-(c\tilde{t})^2 + \tilde{x}^2 = L^2$ through the point $A$ lies to the right of the line $\tilde{x} = L$, touching only at the point of tangency at $A$. As such, the event $B$ which occurs on $\tilde{x} = L$ is between the origin $O$ and the point $C$ along the line $t = 0$ which intersects the hyperbola. Thus $C$ must have coordinate $(t, x) = (0, L)$, since it lies on the hyperbola. Hence the point $B$ must have $x$-coordinate *less than* $L$.

1.2.3.1. *Pole and Barn Paradox.* Consider a barn of rest length 10, and a pole of rest length 20. Suppose they are in inertial frames moving along the $x$-axis with respect to each other with relative velocity $v$, $\frac{v}{c} = \frac{\sqrt{3}}{2}$, so that $\sqrt{1-(v/c)^2} = \frac{1}{2}$. From the point of view of the barn, the pole contracts along the direction of motion to half its rest length, so that it can fit entirely in the barn as it moves through. From the point of view of the pole, the barn is moving toward it, and thus it contracts to length 5; thus in the rest frame of the pole, it can never fit entirely inside the barn. Can both viewpoints of reality be correct?

The answer of course is *Yes*! Let's analyze this first from the point of view of the barn frame $\mathcal{O}$, in which the one end of the barn has world-line $x = 0$, and the other end $x = 10$. The pole is moving in the positive $x$-direction, and at time $t = 0$ in $\mathcal{O}$, the ends of the pole are at $x = 0$ (front) and $x = -10$ (back). The world-lines for the front and back ends of the pole are respectively $x = \frac{v}{c}t = \frac{\sqrt{3}}{2}\,t$ and $x = -10 + \frac{\sqrt{3}}{2}\,t$. At time $t = \frac{20}{\sqrt{3}}$, the front end of the pole is at $x = 10$ (let's call this point in space-time $A$, and the back end at $x = 0$ (this point in space-time is $B$).

From the pole frame $\tilde{\mathcal{O}}$, the barn never contains the pole. That the two observers disagree is not a paradox, since simultaneity is relative. Indeed, the issue is simply that in $\tilde{\mathcal{O}}$, the events $A$ and $B$ are not simultaneous like they are in $\mathcal{O}$. This is easy to compute by using the Lorentz transformations on the points with coordinates $(t, x) = (\frac{20}{\sqrt{3}}, 10)$ (point $A$) and $(t, x) = (\frac{20}{\sqrt{3}}, 0)$ (point $B$).

**1.2.4. Acceleration.** If $\gamma(\tau)$ is a future-pointing time-like curve which is parametrized by proper time $\tau$, then for any $\tau$, there is an inertial frame (coordinate chart) for Minkowski space for which $\gamma(\tau)$ corresponds to the origin, and $\gamma'(\tau) = \frac{\partial}{\partial t}$. Such a coordinate chart is called a *momentarily co-moving rest frame*. Since $\langle \gamma'(\tau), \gamma'(\tau)\rangle$ is constant, then the (covariant) acceleration $\gamma''(\tau) = \frac{D}{d\tau}\alpha'(\tau)$ is orthogonal to $\gamma'(\tau)$. In a co-moving rest frame, then, $\gamma''(\tau)$ is purely spatial. Thus $\langle \gamma''(\tau), \gamma''(\tau)\rangle \geq 0$.

1.2.4.1. *Constant acceleration.* Suppose $\gamma(\tau)$ is a future-pointing time-like curve parametrized by proper time $\tau$, so that $\langle \gamma''(\tau), \gamma''(\tau)\rangle$ equals a constant $a^2$ for some $a > 0$. Then in fact we can specify $\gamma(\tau)$ by finding its parametrization $(t(\tau), x(\tau))$ in an inertial frame in which $\gamma'(0) \in \operatorname{span}\left\{\frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right\}$. In this case, $y'(0) = z'(0) = 0$, and since the acceleration condition implies $y''(\tau) = 0 = z''(\tau)$, we thus have that $y(\tau)$ and $z(\tau)$ are constant. As for the $t$ and $x$ coordinates, we have the two conditions

$$c^2\left(\frac{dt}{d\tau}\right)^2 - \left(\frac{dx}{d\tau}\right)^2 = c^2$$
$$-c^2\left(\frac{d^2t}{d\tau^2}\right)^2 + \left(\frac{d^2x}{d\tau^2}\right)^2 = a^2.$$

Since $\frac{dt}{d\tau} > 0$ ($\gamma$ is future-pointing), we can find a smooth function $f(\tau)$ so that $\frac{dt}{d\tau} = \cosh(f(\tau))$ and then $\frac{dx}{d\tau} = c\sinh(f(\tau))$. Inserting into the second equation, we obtain

$$c^2(f'(\tau))^2 = a^2.$$

Therefore $f(\tau) = \pm(\frac{a}{c}\tau + \tau_0)$. Hence by integration we obtain

$$x^0(\tau) := ct(\tau) = \frac{c^2}{a}\sinh(\frac{a}{c}\tau + \tau_0) + t_0$$
$$x(\tau) = \pm\frac{c^2}{a}\cosh(\frac{a}{c}\tau + \tau_0) + x_0.$$

Thus the curve is given in the $x^0x$-plane (equivalently $tx$-plane) as a hyperbola.

1.2.4.2. *Trying to catch up to light.* We finish this section with an interesting example. Built into Special Relativity is the feature that all inertial observers measure the same speed of light in vacuum. However, one might wonder if by accelerating one's motion, one could do better and catch up with light, at least to some extent. So, consider for $b > 0$ a curve $\gamma(\tau)$ with uniform acceleration $a = bc$, whose components in an inertial frame are given by $t(\tau) = \frac{1}{b}\sinh(b\tau)$, $x(\tau) = \frac{c}{b}\cosh(b\tau)$. In the $tx$-plane the path forms a hyperbola, and as one can infer from a sketch of the trajectory, any light signal sent from along $\gamma$ will reach the inertial observer at $x = 0$ at some time $t > 0$, but if reflected, will *never* reach $\gamma$. Moreover, suppose that at $\gamma(0)$, a photon of light is emitted. $\gamma$ is accelerating and moving in the positive $x$ direction for $\tau > 0$, so what speed does $\gamma$ measure for the photon? Well, since $\gamma(\tau)$ is orthogonal to $\gamma'(\tau)$, the ray from the origin through $\gamma(\tau)$ is comprised of events which $\gamma$ will measure as simultaneous with $\gamma(\tau)$. If we consider the path of the photon as parametrized by $(t(s), x(s)) = (s, \frac{c}{a} + cs)$, then it intersects this ray at some point $A_\tau = h(\tau)\gamma(\tau)$. Thus $s = \frac{h(\tau)}{b}\sinh(b\tau)$, so that

$$\frac{c}{a} + c\frac{h(\tau)}{b}\sinh(b\tau) = \frac{c}{b}\cosh(b\tau),$$

from which we see $h(\tau) = e^{b\tau}$. The displacement vector from $\gamma(\tau)$ to $A_\tau$ is space-like, and its length is the distance from $\gamma(\tau)$ to the photon as measured after $\tau$ units of time have elapsed as measured along $\gamma$. This displacement has length $|(e^{b\tau} - 1)\ \overrightarrow{O\gamma(\tau)}\ | = \frac{c}{b}(e^{b\tau} - 1)$. This is the distance from $\gamma(\tau)$ to the photon, the derivative with respect to $\tau$ of which is $ce^{b\tau}$. So, not only is the acceleration not helping the observer $\gamma$ make any progress on catching the photon, but as measured along $\gamma$, the photon is accelerating *away* from $\gamma$. This is counterintuitive, but is a consequence of how $\gamma$ makes measurements along its world-line, and furthermore, keep in mind that in any momentarily co-moving rest frame, the photon has speed $c$.

**1.2.5. Electromagnetism.** We briefly consider how the electromagnetic fields transform under a Lorentz transformation. As we saw earlier, we expect the electric and magnetic fields to somehow transform together, since charges in motion induce and are affected by a magnetic field, but motion is relative. In fact the fields transform together as an anti-symmetric two-tensor called the *Faraday tensor*. In an inertial system $\mathcal{O}$ (and using $x^0 = ct$), where the electric field has components $E^i$ and the magnetic field $B^i$, the Faraday tensor has a component matrix

$$(F^{\mu\nu}) = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & B^3 & -B^2 \\ -E^2 & -B^3 & 0 & B^1 \\ -E^3 & B^2 & -B^1 & 0 \end{pmatrix}.$$

We can write $F$ as a two-form (with components down) as (where we take $E_i = E^i$, $B_i = B^i$ in inertial coordinates)

$$F^\flat = (E_1 dx^1 + E_2 dx^2 + E_3 dx^3) \wedge dx^0 + (B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2).$$

How do the fields transform exactly? For example, let's compute the magnetic field component $\tilde{B}^3$ in another inertial frame $\tilde{\mathcal{O}}$ moving along the $x$-axis at velocity $v$ relative to $\mathcal{O}$. If we let the Lorentz transformation have matrix $\Lambda^\mu_\nu$ given by

$$[\Lambda^\mu_\nu] = \begin{pmatrix} \frac{1}{\sqrt{1-(v/c)^2}} & -\frac{v/c}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ -\frac{v/c}{\sqrt{1-(v/c)^2}} & \frac{1}{\sqrt{1-(v/c)^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

then (using the Einstein convention of summing over repeated upper and lower indices)

$$\begin{aligned} \tilde{B}^3 = \tilde{F}^{12} = \Lambda^1_{\ \mu} F^{\mu\nu} \Lambda^3_{\ \nu} &= \Lambda^1_{\ 0} F^{02} \Lambda^2_{\ 2} + \Lambda^1_{\ 1} F^{12} \Lambda^2_{\ 2} \\ &= -\frac{v/c}{\sqrt{1-(v/c)^2}} E^2 + \frac{1}{\sqrt{1-(v/c)^2}} B^3 \\ &= \frac{1}{\sqrt{1-(v/c)^2}} \Big( B^3 - \frac{v}{c} E^2 \Big). \end{aligned}$$

$$\tilde{E}^1 = \tilde{F}^{01} = \Lambda^0_{\ \mu} F^{\mu\nu} \Lambda^1_{\ \nu} = \Lambda^0_{\ 0} F^{01} \Lambda^1_{\ 1} + \Lambda^0_{\ 1} F^{10} \Lambda^1_{\ 0} = E^1.$$

One can check the other transformation rules:

$$\begin{aligned} \tilde{B}^1 &= B^1 \\ \tilde{B}^2 &= \frac{1}{\sqrt{1-(v/c)^2}} \Big( B^2 + \frac{v}{c} E^3 \Big) \\ \tilde{E}^2 &= \frac{1}{\sqrt{1-(v/c)^2}} \Big( E^2 - \frac{v}{c} B^3 \Big) \\ \tilde{E}^3 &= \frac{1}{\sqrt{1-(v/c)^2}} \Big( E^3 + \frac{v}{c} B^2 \Big). \end{aligned}$$

In a region with electromagnetic fields but free of charges and hence current (for simplicity), two of Maxwell's four equations are captured by $\mathrm{div}F = 0$, i.e. $F^{\mu\nu}_{;\nu} = 0$. In inertial coordinates, the covariant derivative is just a partial derivative, and we see for $\mu = 0$, we get Gauss' Law div $\mathbf{E} = 0$ (spatial divergence used here), while the other components give $\frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} =$ curl $\mathbf{B}$.

Where do the other two Maxwell equations come from? Well, compute $dF^\flat = 0$, and you find this is equivalent to div $\mathbf{B} = 0$ and $\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} = -$curl $\mathbf{E}$ (note $\frac{\partial}{\partial x^0} = \frac{1}{c}\frac{\partial}{\partial t}$).

Maxwell's equations in any inertial frame will take the same form—in any inertial frame the Christoffel symbols vanish and the metric has the same components. As *tensorial* equations, we can compute the form of Maxwell's equations in any non-inertial coordinate system too, of course.

## 1.3. Energy and Momentum

We begin by a thought experiment due to Einstein. Imagine a box (and its contents) of total mass $M$ and length $L$, at rest. Suddenly from inside the left side of the box some photons of total energy $E$ are emitted in the direction toward the right side of the box. The formula for photon momentum $p$ is $E = cp$. By conservation of momentum, then, the box should acquire a net momentum $-p = Mv$ (to the left). For $|v| << c$, the time $\Delta t$ for the photons to get to the right side of the box is about $\Delta t \approx \frac{L}{c}$. In this time, the box moves to the left, and $\Delta x = v\Delta t \approx \frac{vL}{c}$. When the photons reach the right side of the box and stop, the motion ceases.

Einstein argues that there is no reason why in this closed system the center of mass should have changed from the start to the end of the process. He suggests that the photons must have carried a mass $m$ to the right side of the box to balance out the center, i.e. $m(L - \Delta x) + M\Delta x = 0$. Thus we obtain $m = \frac{-M\Delta x}{L-\Delta x} = \frac{-M\Delta x}{L} + O(|\Delta x|^2) \approx \frac{-Mv}{c} = \frac{E/c}{c}$, or

$$E = mc^2.$$

This is the famous formula relating (rest) mass $m$ to energy.

### 1.3.1. Energy-momentum four-vector.

Associated to any massive particle is a quantity $m$ called its *rest mass*. The rest mass is a measure of *inertia*, and one can imagine measuring the inertial mass by applying Newton's laws in a frame of reference (coordinate chart) in which the massive particle is (momentarily) at rest. A detailed discussion of the technicalities involved in the physical concept would take us too far afield for now. We just note that the rest mass corresponds to the mass one uses in Newton's Second Law $\mathbf{F} = m\mathbf{a}$ in classical mechanics. Of course, special relativity re-writes the kinematical and dynamical laws, but there is still an analogous concept of mass. The term *rest mass* indicates that in relativity, the quantity *mass* may itself be relative. However, the rest mass is by definition an invariant, since it is associated to the object itself, as measured in a frame adapted to the object.

If $\gamma(\tau)$ is a time-like curve parametrized by proper-time which represents the word-line of the particle, then the *energy-momentum four-vector* is $\mathbb{P} = m\gamma'(\tau)$. In a momentarily co-moving rest-frame $\tilde{\mathcal{O}}$ at a point on $\gamma$ (i.e. $\gamma'(\tau) = \frac{\partial}{\partial \tilde{t}}$ at this point in this frame), $\mathbb{P} = m\frac{\partial}{\partial \tilde{t}} = mc\frac{\partial}{\partial \tilde{x}^0}$, where $\tilde{x}^0 = c\tilde{t}$. Note that $-\langle \mathbb{P}, \mathbb{P} \rangle = (mc)^2 = (E_0/c)^2$, where $E_0 = mc^2$ is the *rest energy.* If instead we now use an inertial frame $\mathcal{O}$ in which the particle is moving with velocity $v$ along the $x$-axis, then by (1.4),

$$\mathbb{P} = \frac{m}{\sqrt{1-(v/c)^2}}\frac{\partial}{\partial t} + \frac{mv}{\sqrt{1-(v/c)^2}}\frac{\partial}{\partial x} = \frac{mc}{\sqrt{1-(v/c)^2}}\frac{\partial}{\partial x^0} + \frac{mv}{\sqrt{1-(v/c)^2}}\frac{\partial}{\partial x}.$$

We identify $\frac{m}{\sqrt{1-(v/c)^2}}c^2 = E$ as the energy of the particle as measured by the inertial frame $\mathcal{O}$ in which the particle is moving, and we identity $\mathbf{p} = \frac{mv}{\sqrt{1-(v/c)^2}}\frac{\partial}{\partial x}$ as the *spatial momentum.* Since the spatial velocity is $\mathbf{v} = v\frac{\partial}{\partial x}$, from the point of view of the inertial frame with respect to which the particle is moving, the mass is given by $m(v) = \frac{m}{\sqrt{1-(v/c)^2}}$. Note that $\lim_{v \nearrow c} m(v) = +\infty$, which indicates that inertia increases with speed; this is consistent with the fact that a constant force cannot accelerate any particle to or above the speed of light. Furthermore, note that by Taylor (binomial) expansion,

$$E = m(v)c^2 = mc^2(1 + \frac{1}{2}(v/c)^2 + O((v/c)^4)) = mc^2 + \frac{1}{2}mv^2 + O(v^4/c^2).$$

The first two terms are the rest energy and the kinetic energy. Furthermore if $\mathbb{U}_{\text{obs}}$ is a time-like unit vector tangent to the path of an observer, then the observed energy of the particle with momentum $\mathbb{P}$ is just $E_{\text{obs}} = -\langle \mathbb{P}, \mathbb{U}_{\text{obs}} \rangle$. Finally, we note

$$E_0^2 = -c^2\langle \mathbb{P}, \mathbb{P} \rangle = (m(v)c^2)^2 - c^2\|\mathbf{p}\|^2 = E^2 - c^2\|\mathbf{p}\|^2.$$

Note that in units where $c = 1$, we have $\mathbb{P} = E\frac{\partial}{\partial x^0} + \mathbf{p}$, and $m^2 = E^2 - \|\mathbf{p}\|^2$.

**1.3.2. The stress-energy tensor.** One of the constituents of the Einstein equation in general relativity is the stress-energy, or energy-momentum tensor, which encodes the energy and momentum fluxes associated to the physical matter or fields in the space-time. This is a generalization to space-time of the spatial stress tensor of classical mechanics. We introduce it here, just enough to motivate its place in the formulation of general relativity.

We define the stress-energy tensor as a $(2,0)$-tensor. Consider a one-form $\theta$ at a point $P$ in space-time (which we take to be four-dimensional here, as usual, though of course we can generalize to other dimensions). We assume that $\theta$ is dual to either a time-like future pointing vector, or a space-like vector. In particular, it has a non-zero metric norm, which we normalize to be $\langle\theta,\theta\rangle = \pm 1$. For example, $\theta = dx^0 = c\ dt$, or $\theta = dx^i$, $i = 1,2,3$. In a region of space-time, we imagine there is a collection of particles, or a physical field, possessing an energy-momentum four-vector field at each point. We consider $\theta \neq 0$ as a linear functional operating on the tangent space at $P$. Its nullspace $W$ is three-dimensional, and by assumption on the causal nature of $\theta$, we see it is either space-like, or time-like (Lorentzian) as a subspace. Let $B \subset W$ be a region about $P$, and consider the net vector sum (integral) $\Delta\mathbb{P}$ of all the energy-momentum four-vectors associated to the particles/fields at points in the space-time region $B$; in the space-like case, when forming the sum, the contribution of any term $\widetilde{\mathbb{P}}$ must be modified for direction to $\operatorname{sgn}(\theta(\widetilde{\mathbb{P}})) \cdot \widetilde{\mathbb{P}}$. This accounts for flow across the corresponding spatial boundary determining the box $B$. Let $\Delta V$ be the volume of $B$ with respect to the metric. Then we define the vector field $T(\theta,\cdot)|_P = \lim\limits_{B\to\{P\}} \frac{c\Delta\mathbb{P}}{\Delta V}$. If $\eta$ is another one-form at $P$, we define $T(\theta,\eta) = \eta(T(\theta,\cdot))$. Now we extend the definition of $T$ by allowing scaling in $\theta$; given $\theta$ which is time-like or space-like, we define $\hat{\theta}$ by $\theta = b\hat{\theta}$, where $|b| = \sqrt{\pm\langle\theta,\theta\rangle}$ and the sign of $b$ is positive unless $\theta$ is past time-like, in which case $b < 0$. We then define $T(\theta,\eta)$ as $bT(\hat{\theta},\eta)$, where the latter quantity has been defined above.

Clearly, $T(\theta,\eta)$ is linear in $\eta$. Physical reasoning as in classical mechanics may be used to argue that $T$ should also be linear in $\theta$. What may be more surprising is that when $\eta$ is also either time-like or space-like, we can argue that $T(\theta,\eta) = T(\eta,\theta)$. We then see that $T$ can be extended to all forms, and yields a *symmetric* $(2,0)$-tensor.

Let us interpret the components of this tensor in some inertial frame. $T(dx^0,\cdot)$ is $c$ times the spatial density of the energy-momentum four-vectors of the physical particles/fields at $P$. Then $T^{00} = T(dx^0,dx^0) = dx^0(T(dx^0,\cdot))$ is precisely the *energy density* $\rho$ at $P$ as measured in this frame. Similarly, for $i = 1,2,3$, $T^{0i} = cT(dt,dx^i)$ is $c$ times the $x^i$-component of the momentum density (momentum per unit spatial volume) at $P$. Similarly, consider $T(dx^1,\cdot)$. The space-time box $B$ is now determined by a rectangle $R$ of area $\Delta A$ in the $x^2x^3$-plane, as well as a side of length $\Delta x^0 = c\Delta t$. Thus $\Delta V = c\Delta t\Delta A$. Thus $T^{10} \sim \frac{\Delta E}{c\Delta t\Delta A}$ equals $\frac{1}{c}$ times the rate of flux of the energy (units of energy per unit area per unit time) in the $x^1$-direction. Note that by symmetry, $T^{10} = T^{01}$, which reflects the equivalence of mass and energy (note how the units work out). $T^{11} \sim \frac{\Delta\mathbf{p}^1}{\Delta t\Delta A}$, and as the force is given by $\mathbf{F} = \frac{d\mathbf{p}}{dt}$, we have that $T^{11}$ is the *normal stress* in the $x^1$-direction, that is, the force per unit area normal to the $x^2x^3$-direction. On the other hand the component $T^{12}$ is then a *shear*, the force per unit area on the $x^2x^3$-plane acting in the $x^2$-direction—this is a component of force *tangential* to the relevant area element. The normal stress is sometimes called the *pressure* when it is independent of direction.

Another key property of a stress-energy tensor is that it is *divergence-free*. To motivate this, we compute a component of the divergence in inertial coordinates, in which the Christoffel symbols vanish. In such a coordinate system, $c$ times the zero-component of the divergence is then $c(T^{00}_{,0} + T^{i0}_{,i}) = \frac{\partial \rho}{\partial t} + cT^{i0}_{,i}$ where $i = 1, 2, 3$, and recall we use the *Einstein summation convention*. As discussed above, $cT^{i0}\frac{\partial}{\partial x^i}$ is the vector field measuring the rate of spatial flux of the energy. Applying the divergence theorem to a spatial region $R$, we obtain the time rate of change of the energy in the region is

$$\frac{\partial}{\partial t}\int_R \rho\, dV = \int_R \frac{\partial \rho}{\partial t}\, dV = \int_R cT^{00}_{,0}\, dV.$$

By conservation of energy, this must balance with the rate of flux of energy *into* the region across the spatial boundary $\partial R$ with *outward* unit normal $\mathbf{n}$, which is given by the divergence theorem as

$$\int_{\partial R} cT^{i0}\frac{\partial}{\partial x^i}\cdot(-\mathbf{n})\, dA = -\int_R cT^{i0}_{,i}\, dV.$$

Since the region $R$ may be made arbitrarily small, we obtain a component of $\operatorname{div}(T) = 0$. The other components may be derived similarly using conservation of momentum.

We now introduce several standard examples of the stress-energy tensor. Of course, when in vacuum (free of fields or particles), $T = 0$. The next simplest example is that of *dust*. Consider a collection of particles which are all at rest in some inertial frame $\tilde{\mathcal{O}}$. In this frame, the energy density alone determines the stress-energy tensor: $T = c^{-2}\rho\frac{\partial}{\partial \tilde{t}} \otimes \frac{\partial}{\partial \tilde{t}} = \rho\frac{\partial}{\partial \tilde{x}^0} \otimes \frac{\partial}{\partial \tilde{x}^0}$. This can be written invariantly as $T = c^{-2}\rho\mathbb{U} \otimes \mathbb{U}$, where $\mathbb{U}$ is the four-velocity of the dust, where $\rho$ is the *rest* energy density of the dust. Using the Lorentz transformation, one can determine the energy density in a frame $\mathcal{O}$ with respect to which the dust moves with velocity $v$, which we align along the $x$-axis for simplicity. Indeed in such a frame, we have by (1.4)

$\mathbb{U} = \frac{\partial}{\partial \tilde{t}} = \frac{1}{\sqrt{1-(v/c)^2}} \frac{\partial}{\partial t} + \frac{v}{\sqrt{1-(v/c)^2}} \frac{\partial}{\partial x} = \frac{c}{\sqrt{1-(v/c)^2}} \frac{\partial}{\partial x^0} + \frac{v}{\sqrt{1-(v/c)^2}} \frac{\partial}{\partial x}$, so that

$$\begin{aligned} T &= c^{-2} \rho \mathbb{U} \otimes \mathbb{U} \\ &= \frac{\rho}{1-(v/c)^2} \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} \\ &\quad + \frac{\rho v/c}{1-(v/c)^2} \left( \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x} + \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x^0} \right) + \frac{\rho (v/c)^2}{1-(v/c)^2} \frac{\partial}{\partial x} \otimes \frac{\partial}{\partial x}. \end{aligned}$$

We note that all the terms include two factors of $\sqrt{1-(v/c)^2}$ in the denominator. In particular the energy density as measured in $\mathcal{O}$ is $\frac{\rho}{1-(v/c)^2}$, which includes one such factor for the transformation of mass, and another one for the length contraction, which affects the value of the measured density in $\mathcal{O}$. Similar analysis holds for the other components.

A *perfect fluid* is a continuum model described by its four-velocity vector field $\mathbb{U}$, as well as its density $\rho$ and pressure $p$. In coordinates which are (momentarily) co-moving with the fluid, there is no shear, and the normal stresses are the same in all directions, namely the pressure. The requirement of zero viscosity (shear) should hold in all (momentarily) co-moving reference frames, and this can only be true if the spatial components of the tensor are given by a multiple of the identity matrix. Furthermore, we note that requiring $T^{0i} = 0$ in such reference frames amounts to requiring there to be no heat conduction in these frames. An invariant way to express the perfect fluid stress-energy tensor is then $T = c^{-2}(\rho + p)\mathbb{U} \otimes \mathbb{U} + p\eta^\sharp$ where $\eta^\sharp$ is the contravariant form of the metric tensor $\eta$ (indices up). In a rest frame of a fluid element, $T = \rho \frac{\partial}{\partial x^0} \otimes \frac{\partial}{\partial x^0} + p\delta^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j}$. For comparison, we note that the dust model is an idealization in which the constituent particles have no random motion at all (they are all at rest in a certain frame), and thus there is no pressure.

Finally, we define the electromagnetic stress-energy tensor as

$$T^{\mu\nu} = \frac{1}{4\pi} \left( F^{\mu\alpha} \eta_{\alpha\beta} F^{\nu\beta} - \frac{1}{4} \eta^{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} \right).$$

Observe that $F_{0i} = -F^{0i}$, so that

$$T^{00} = \frac{1}{4\pi} \left( F^{0\alpha} \eta_{\alpha\beta} F^{0\beta} - \frac{1}{4}(-2|\mathbf{E}|^2 + 2|\mathbf{B}|^2) \right) = \frac{1}{8\pi}(|\mathbf{E}|^2 + |\mathbf{B}|^2),$$

the familiar formula for the energy density of the electromagnetic field (in cgs units). For later reference, we also note that $T^{0i} = (\mathbf{E} \times \mathbf{B})^i$.

## 1.4. Some geometric aspects of Minkowski space-time

In this section we again use inertial coordinates $(x^0, x^1, x^2, x^3)$ for Minkowski space $\mathbb{M}^4 = \mathbb{R}^4_1$, so that the metric is just $-(dx^0)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2$, and analogously for $\mathbb{M}^{1+k} = \mathbb{R}^{1+k}_1$. We will blur the distinction between a point $P$ and its coordinates $x^\mu(P)$. We let $D$ be the connection compatible with the Minkowski metric.

**1.4.1. Hyperquadrics.** We consider the level sets of the function $F(x) = -(x^0)^2 + (x^1)^2 + (x^2)^2 + \cdots + (x^k)^2$. The level set $\Sigma = F^{-1}(0)$ is precisely the set of all points whose position vector from the origin $O$ is null; on other words, $\Sigma$ is the light cone from the origin $O$. $\Sigma \setminus \{O\} = \Sigma^+ \cup \Sigma^-$ is a smooth *null* hypersurface: at any point $P \in \Sigma \setminus \{O\}$ with coordinates $(x^\mu(P))$, the null vector $x^\mu \frac{\partial}{\partial x^\mu}$ (summation convention in force!) is *both* tangent *and* normal to $\Sigma$ at $P$.

1.4.1.1. *Hyperbolic space.* We now consider the smooth hypersurface $\Sigma$ equal to one of the two components of $F^{-1}(-r^2)$, for $r > 0$, say the component where $x^0 > 0$. The vector $n = r^{-1} x^\mu \frac{\partial}{\partial x^\mu}$ is a unit *time-like* normal vector to $\Sigma$. The induced metric on the hypersurface is thus Riemannian. If $Y = Y^\mu \frac{\partial}{\partial x^\mu}$ is tangent to $\Sigma$, then

$$D_Y n = r^{-1} D_Y(x^\mu) \frac{\partial}{\partial x^\mu} = r^{-1} Y^\mu \frac{\partial}{\partial x^\mu} = r^{-1} Y.$$

Thus the second fundamental form $\mathit{I\!I}$ of $\Sigma$ is then just

$$\mathit{I\!I}(Y, Z) = \langle D_Y Z, n\rangle \langle n, n\rangle n = \langle D_Y n, Z\rangle n = r^{-1} \langle Y, Z\rangle n.$$

Thus $\Sigma$ is totally umbilic. We can compute its curvature via the Gauss equation (see Proposition 3.5): for $X, Y, Z, W$ tangent to $\Sigma$,

$$\begin{aligned} \langle R(X,Y,Z), W\rangle &= \langle R^\Sigma(X,Y,Z), W\rangle - \langle \mathit{I\!I}(X,Z), \mathit{I\!I}(Y,W)\rangle \\ &\quad + \langle \mathit{I\!I}(X,W), \mathit{I\!I}(Y,Z)\rangle. \end{aligned} \tag{1.10}$$

Thus with $R = 0$ on Minkowski space, we insert the formula for the second fundamental form to obtain

$$\langle R^\Sigma(X,Y,Z), W\rangle = r^{-2} \left( \langle X, Z\rangle \langle Y, W\rangle \langle n, n\rangle - \langle X, W\rangle \langle Y, Z\rangle \langle n, n\rangle \right). \tag{1.11}$$

If $E_1, E_2$ is an orthonormal frame on $\Sigma$ spanning a two-plane $\Pi$ tangent to $\Sigma$, then

$$K(\Pi) = \langle R^\Sigma(E_1, E_2, E_1), E_2\rangle = -r^{-2}.$$

Thus $\Sigma$ has constant negative sectional curvature. Since $\Sigma$ is simply connected, $\Sigma$ is isometric to the hyperbolic space of curvature $-r^{-2}$. For $r = 1$, then $\Sigma \subset \mathbb{M}^4$ is isometric to $\mathbb{H}^3$. In $\mathbb{M}^{1+k}$ with $k \geq 2$, the set $F^{-1}(-1)$ is isometric to unit hyperbolic space $\mathbb{H}^k$.

1.4.1.2. *De Sitter space-time.* The level set $F^{-1}(r^2) =: \mathbb{S}^k_1(r)$ for $r > 0$ is a smooth connected hypersurface which is topologically $\mathbb{R} \times \mathbb{S}^{k-1}$, so that for $k \geq 3$, $\mathbb{S}^k_1(r)$ is simply connected. $\mathbb{S}^4_1(1)$ is called *de Sitter space-time.* In this case the vector $n = r^{-1}x^\mu \frac{\partial}{\partial x^\mu}$ is a unit *space-like* normal vector to the level set. The induced metric on $\mathbb{S}^k_1(r)$ is *Lorentzian*: the vector $(r^2 + (x^0)^2)\frac{\partial}{\partial x^0} + x^0 x^i \frac{\partial}{\partial x^i}$ is time-like and tangent to the submanifold at a point $P$ given by $(x^\mu(P))$. If $\Pi$ is a space-like two-plane tangent to $\mathbb{S}^k_1(r)$, then (1.11) implies $K(\Pi) = r^{-2}$. Recall that the sectional curvature of a non-degenerate two-plane $\Pi$ spanned by $V$ and $W$ is

$$K(\Pi) = \frac{\langle R(V, W, V), W\rangle}{\langle V, V\rangle\langle W, W\rangle - \langle V, W\rangle^2}.$$

So even for a plane non-degenerate plane $\Pi$ spanned by orthonormal vectors $E_0$ (time-like) and $E_1$ (space-like), we still have $K(\Pi) = r^{-2}$.

**1.4.2. Conformal Compactification of $\mathbb{M}^4$.** We now present the classical compactification of $\mathbb{M}^4$. The motivation was to find a way to topologically compactify the space-time in a way that preserved the causal structure, and in particular preserved the null cones. In this way, one can faithfully represent the null geometry on a compact region, whose boundary in part represents the null paths *at infinity.* This can be generalized to $\mathbb{M}^{1+k}$, and has been used to prove the existence of global solutions to quasilinear hyperbolic systems with "small" initial data (cf. D. Christodoulou, CPAM **39**, no. 2, 267-282 (1986)).

We will obtain the compactification by applying two coordinate changes. The first is to introduce advanced and retarded null coordinates $v = x^0 + r$, $u = x^0 - r$, where $r^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$. The Minkowski metric becomes

$$-du\, dv + \frac{1}{4}(v - u)^2 g_{\mathbb{S}^2},$$

where $du\, dv = \frac{1}{2}(du \otimes dv + dv \otimes du)$, and $g_{\mathbb{S}^2}$ is the metric on the round unit sphere. Note that the level sets of $u$ and $v$ are null. If one holds $u$ and a point $\omega \in \mathbb{S}^2$ fixed, then varying $v \to +\infty$ corresponds to going forward in time along a null geodesic, to infinity. Likewise, with $v$ fixed, $u \to -\infty$ corresponds to a path of light going to past infinity. The goal is to represent "where these null paths go at infinity." One way to do this is to use the inverse tangent function to define new coordinates

$$T = \tan^{-1}(v) + \tan^{-1}(u)$$
$$R = \tan^{-1}(v) - \tan^{-1}(u).$$

Since $dT = \frac{1}{1+v^2}\, dv + \frac{1}{1+u^2}\, du$ and $dR = \frac{1}{1+v^2}\, dv - \frac{1}{1+u^2}\, du$, we can easily derive the Jacobian determinant

$$\frac{\partial(T,R)}{\partial(u,v)} = \frac{2}{(1+v^2)(1+u^2)} > 0.$$

Moreover,

$$-dT^2 + dR^2 = \frac{4}{(1+v^2)(1+u^2)}(-du\, dv) =: \Omega^2(-du\, dv).$$

Note that $\Omega^2 = \frac{4}{(1+t^2+r^2)^2-4t^2r^2}$ is smooth on all of $\mathbb{M}^4$, and that

$$\begin{aligned}\sin R &= \sin(\tan^{-1}(v))\cos(\tan^{-1}(u)) - \sin(\tan^{-1} u)\cos(\tan^{-1}(v)) \\ &= \frac{v-u}{\sqrt{(1+v^2}\sqrt{1+u^2}}.\end{aligned}$$

Thus $\sin^2 R = \frac{1}{4}\Omega^2(v-u)^2$. This implies

$$-dT^2 + dR^2 + \sin^2 R\, g_{\mathbb{S}^2} = \Omega^2\left(-du\, dv + \frac{1}{4}(v-u)^2 g_{\mathbb{S}^2}\right).$$

In other words, the metric on the left, which is easily identified as a Lorentzian product metric on $\mathbb{R}\times\mathbb{S}^3$ (called the *Einstein static universe*), is *conformal* to the Minkowski metric. In other words, using the coordinate change, we produce an embedding of Minkowksi space-time into the Einstein static universe, which is not an isometric, but a *conformal isometry.* Thus it preserves the causal nature of vectors, in particular the null structure. The image of the embedding is a bounded set: indeed, note that $-\pi < T < \pi$ and $0 \le R < \pi$ on Minkowski space-time. The boundary of the set is the union of two smooth null hypersurfaces $\mathcal{J}^\pm$, "scri-plus" and "scri-minus", where "scri" is short for "script 'I'." We note that $\Omega$ is a defining function for $\mathcal{J}^\pm$, since $\Omega = 0$ here, with $d\Omega \neq 0$.

We now briefly describe some of the features of the boundary. Note that $T+R = 2\tan^{-1}(v)$ and $T-R = 2\tan^{-1}(u)$. Thus the null rays to the future end up ($v\to+\infty$) at $T+R=\pi$, which for $0<R<\pi$ gives $\mathcal{J}^+$; similarly for $T-R=\pi$ ($u\to-\infty$) and $\mathcal{J}^-$. The null vector $\frac{\partial}{\partial T} \mp \frac{\partial}{\partial R}$ is both tangent *and* normal to $\mathcal{J}^\pm$. The closure of the image of Minkowski space can be represented by a $T$-$R$ triangle, bounded by $R=0$, and $T\pm R = \pm\pi$. Every point in this region represents a two-sphere, except where $R=0$ or $R=\pi$, each point of which represents a point. One can argue that time-like geodesics must start at $i^-$ in the past, corresponding to $(T,R) = (-\pi,0)$ and must end at $i^+$ corresponding to $(T,R) = (\pi,0)$. We let $i^0$ be the point corresponding to $(T,R) = (0,\pi)$, which is called *space-like infinity.* Space-like curves with $r\to+\infty$ have a limit at $i^0$ in the conformal picture.

*Chapter 2*

# The Einstein Equation

The theory of special relativity incorporates a modification of Newtonian dynamics together with electromagnetism. A natural question to consider is how gravitation fits into the framework of relativity. Analysis of this question will lead us to the Einstein equation. We begin, however, by reviewing Newton's Law of Gravitation.

## 2.1. Newtonian gravity

Newton's Law of Gravity can be formulated as follows. If two objects are separated by a spatial distance $r$, then the magnitude of the gravitational force between them is given by $F = \frac{Gm_gM_g}{r^2}$, where the direction is along the line from one mass to the other. Here $m_g$ and $M_g$ are the *gravitational masses* associated to the two objects, and $G$ is Newton's gravitational constant. If $\hat{\mathbf{r}}$ is the unit vector from the object of mass $M_g$ to the other object, then the force on the object of mass $m_g$ is

$$(2.1) \qquad \mathbf{F} = -\frac{Gm_gM_g}{r^2}\hat{\mathbf{r}} = m_g\nabla\left(\frac{GM_g}{r}\right) = -m_g\nabla\Phi,$$

where $\Phi := -\frac{GM_g}{r}$ is the *gravitational potential* associated to the mass $M_g$. In analogy with *Coulomb's Law* of electrostatics, namely that the electric force between charged particles (with charges $q_1$ and $q_2$) has magnitude $F = \frac{q_1q_2}{r^2}$ (in cgs units), $m_g$ and $M_g$ play the role of *gravitational charges*. Of course, there is already the notion of mass, or *inertia*, as embodied in Newton's Second Law $\mathbf{F} = m_i\mathbf{a}$, where $m_i$ is the *inertial mass*. By equating forces, we solve for the acceleration of an object of gravitational mass $m_g$ and inertial mass $m_i$ due to the gravitational force of an object of gravitational mass $M_g$:

$$\mathbf{a} = -\frac{m_i}{m_g}\nabla\Phi.$$

From this equation we could discern the ratio of the inertial to the gravitational mass for various objects. It turns out that the acceleration is the same for all bodies, and hence the mass ratio is *constant*, a result epitomized by the famous experiments by Galileo dropping objects of different masses from the tower in Pisa. By adjusting $G$, we may assume, then, that $m_i = m_g = m$: the inertial and gravitational masses agree. The effect of gravity is *universal*: it accelerates all objects the same way, independent of what precisely comprises the mass. In this way gravity is decidedly different from electromagnetism.

Before we move on, we note that the potential function for Newtonian gravity satisfies a simple partial differential equation. Indeed, away from $r = 0$, $\Phi = -\frac{GM}{r}$ is harmonic (with respect to the Euclidean metric), i.e. $\Delta\Phi = 0$, as you can easily check. Of course, $\Delta\Phi$ can be interpreted globally as a distribution, say $T = \Delta\Phi$, and we obtain the equation

$$\Delta\Phi = 4\pi GM\delta_0,$$

where $\delta_0$ is the Dirac measure at the origin (the "location" of the mass $M$). Indeed suppose $\psi \in C_c^\infty(\mathbb{R}^3)$ is any smooth function of compact support, say $\psi(x) = 0$ for $|x| > R$. Then since $\frac{1}{r} \in L^1_{\text{loc}}(\mathbb{R}^3)$, we have by Gauss' Divergence Theorem, Green's identity $\operatorname{div}(\Phi\nabla\psi - \psi\nabla\Phi) = \Phi\Delta\psi - \psi\Delta\Phi$, $\psi = 0$ and $\nabla\psi = 0$ on $\{|x| = R\}$, and $\Delta\Phi = 0$ on $\mathbb{R}^3 \setminus \{O\}$, that

$$\begin{aligned}
T(\psi) := \int\limits_{\mathbb{R}^3} \Phi\Delta\psi \, dV &= \lim_{\epsilon\to 0^+} \int\limits_{\epsilon\le|x|\le R} \Phi\Delta\psi \, dV \\
&= -\lim_{\epsilon\to 0^+} \int\limits_{|x|=\epsilon} (\Phi\nabla\psi - \psi\nabla\Phi)\cdot\hat{\mathbf{r}} \, dA \\
&= \lim_{\epsilon\to 0^+} \int\limits_{|x|=\epsilon} \psi\frac{GM}{r^2}\hat{\mathbf{r}}\cdot\hat{\mathbf{r}} \, dA \\
&= 4\pi GM \lim_{\epsilon\to 0^+} \frac{1}{4\pi\epsilon^2} \int\limits_{|x|=\epsilon} \psi \, dA \\
&= 4\pi GM\psi(0)
\end{aligned}$$

where the last identity follows by continuity. We note that we used the fact that $|\Phi\nabla\psi| \le \frac{C}{r}$ on $\mathbb{R}^3 \setminus \{O\}$, so that $\int\limits_{|x|=\epsilon} \Phi\nabla\psi\cdot\hat{\mathbf{r}} \, dA = O(\epsilon)$.

In this case the matter density is $\sigma = M\delta_0$. For a more general matter distribution of density $\sigma$, the gravitational potential solves Poisson's equation

$$\Delta\Phi = 4\pi G\sigma. \tag{2.2}$$

If $\sigma$ is compactly supported (or more generally, if $\sigma$ decays sufficiently at infinity) then we may choose $\Phi(x) \to 0$ as $|x| \to \infty$.

From the point of view of relativity, however, Newton's Law of Gravitation has a serious flaw, which Newton himself critiqued. If a mass moves, then its change in position will *instantly* effect the gravitational field everywhere else. This means that gravitational effects have infinite speed of propagation. This "action at a distance" bothered Newton and others, and it also is not consistent with causality in special relativity. Einstein sought to rectify this, and in doing so to incorporate another fundamental force within a relativistic framework.

## 2.2. From The Equivalence Principle to General Relativity

In Minkowski space there is a preferred set of coordinate charts, corresponding in physics terminology to *inertial observers.* In such charts the metric for Minkowski space has the familiar form $\eta_{\mu\nu}dx^\mu dx^\nu = -(dx^0)^2 + \delta_{ij}dx^i dx^j$, and any two such charts are related by a Lorentz transformation. These charts are the analogues of Cartesian coordinate systems for Euclidean space $\mathbb{E}^n$.

From the point of view of physics, coordinates from an inertial chart correspond to measurements made by an inertial observer. An interesting physics question, then, is whether inertial observers exist in principle when a gravitational field is present. Einstein applied a now famous thought experiment to this question. Namely, suppose there were an inertial frame of reference, say a small room isolated from other forces or fields. In such a frame, if one lets go of a ball, it would tend to stay at rest. Now suppose on the top of the room, there is attached a rocket, which now accelerates the room "upward" at a uniform rate. If one lets go of a ball now, it will "fall" toward the floor, just like it would in a uniform gravitational field. (Strictly speaking, we are working in a lab frame which is limited in extent in both space and time, so that we can reliably expect a gravitational field to be approximately uniform, i.e. so that it generates a constant acceleration due to gravity. In regions more extended in space-time, the non-uniformity of the gravitational field will give rise to *tidal forces* that can be used to distinguish between uniform acceleration and a gravitational field.) In this case, one cannot distinguish from making local measurements in the lab whether the frame is non-inertial, or whether there is a uniform gravitational field.

Note that this depends on the universality of gravity: it imparts the same acceleration to all objects. This version of the principle of equivalence implies the equality of gravitational and inertial masses: if they were different, then one could distinguish between the two situations by making appropriate measurements. The equality of the gravitational and inertial masses is clearly a direct reflection of the equivalence of a uniform accelerating frame with an inertial frame in which a uniform gravitational field is present.

We have just seen how to create a gravitational field via acceleration. On the flip side, consider the room as an elevator, or even the compartment of one of those amusement park "free fall" rides. In free fall, when the gravitational force is acting, if one lets go of a small object (not safe to try this on the free fall ride!), then its coordinates with respect to the room are constant; this is because a uniform gravitational field accelerates all objects the same. The observer in free fall will then *not* detect a uniform gravitational field. We on the earth claim to detect such a field precisely because we are *not* in free fall: the contact forces on the earth keep us from a free fall path, and thus if we drop a ball, or Newton's apple, then we observe it "fall" to the earth under the force of gravity. This is equivalent to the accelerated room: we "feel" the contact force between the floor and our legs, and we observe objects falling toward the ground. Even light cannot escape "gravity's" pull: if we imagine a light ray entering the room at one end moving toward the other end in a straight line in the inertial frame, then the path of the light is curved in the accelerated frame of reference. Einstein reasoned that by equivalence, a gravitational field should affect the paths of light rays too. Then, could there be an inertial reference frame, where any acceleration not accounted for by contact forces would be attributed to a gravitational force, and in which light rays do not move along straight lines?

Putting that question aside for a moment, we note that the local equivalence between acceleration and gravitation places a roadblock in an attempt to incorporate gravity into the global Minkowski geometry of special relativity. Indeed, imagine two rockets moving with the same uniform acceleration along the same line, one following the other at a distance $\Delta y$. A photon of wavelength $\lambda_0 = \frac{c}{\nu_0}$ (frequency $\nu_0$) travels from the trailing rocket to the lead rocket. In the time it travels, the rockets have a change in velocity $\Delta v$, which we arrange to satisfy $|\Delta v| \ll c$. The photon has a Doppler shift: if $\nu_1$ is the frequency of the received photon, then $\frac{\nu_1}{\nu_0} = \frac{1-\Delta v/c}{\sqrt{1-(\Delta v/c)^2}}$, which to first order in $\frac{\Delta v}{c}$ gives $\frac{\Delta\lambda}{\lambda_0} \approx \frac{\Delta v}{c} \approx \frac{a\Delta y}{c^2}$.

**Exercise 2.1.** Derive the Doppler shift using Lorentz transformations. In the $tx$-plane of an inertial observer, emit pulses one after the other at $x = 0$, first at $t = 0$ then at $t = \Delta t_0$, where $\Delta t_0 = \frac{1}{\nu_0}$ is the period. A moving observer which is at position $x_0$ at $t = 0$ moves with velocity $v > 0$, say, with respect to this inertial frame. If the moving observer receives the first pulse at $(x_1, t_1)$ and the second at $(x_2, t_2)$ in the original frame, then $x_1 = ct_1 = x_0 + vt_1$, and $x_2 = ct_2 = x_0 + vt_2$. Use a Lorentz transformation to convert to the frame of the moving observer.

The above situation should be equivalent to the situation in which there is a uniform gravitational field, with corresponding acceleration $-a$ due to gravity, and a photon travels a "height" $\Delta y$ in the gravitational field—we assume the gravitational field is time-independent, and only varies in $y$. As the change in velocity is the time integral of acceleration, and the equivalent acceleration of the frame to produce the gravitational effect is the gradient $\nabla\Phi$ of the potential $\Phi$, the percentage Doppler shift is approximately given by $c^{-2}(\Phi(y+\Delta y) - \Phi(y))$. If we now try to reconcile this with the Principle of Relativity, we run into trouble. For suppose we have two inertial observers we measure this gravitational field, related to each other by a simple shift of origin: one observer is located at height $y$ and the other at $y + \Delta y$ in the gravitational field. If the observer at $y$ (akin to the trailing rocket) sends a photon of frequency $\nu_0$ to the other observer, the time between the beginning and end of a wavelength is $\Delta t_0 = c^{-1}\lambda_0$. The other observer measures the time between receiving the beginning and end of a wavelength as $\Delta t_1 \approx \Delta t_0 + c^{-1}\Delta\lambda > \Delta t_0$, since the wavelength of the absorbed photon is *longer*—the frequency, and hence the energy, is lower. This means that clocks run at different rates in different places in a gravitational field (confirmed experimentally). However, this should not be the case for these two inertial observers who are not in motion relative to teach other. The space-time paths of the beginning and tail end of the wave should be congruent (the gravitational field only depends on $y$), so that the $\Delta t$ measurements should be the same at the two different heights.

The universality of gravity poses the conundrum that it seems there could not be purely inertial observers in a region of space-time where gravity is acting. Since gravity affects all observers equally, one could not set up background frames of reference from which to measure the motion of a test particle to measure gravity; this is different than electromagnetism, for instance, where neutral particles and charged particles can be distinguished. If we imagine a Newtonian inertial frame far away from massive objects, and then "dial up" a gravitational field, then what happens? All free particles, including light, would be accelerated away from the straight line paths in this coordinate frame. If we actually think about a frame in terms of an observer, however, the universality of gravity implies that the observer would also be affected by gravity. Maybe the problem with the preceding paragraph is that the two observers should have been falling freely in the gravitational field to have been construed to be inertial, as opposed to remaining stationary and measuring the gravitational field. If the observer builds coordinates adapted to the motion (normal or Fermi coordinates along the world-line, see below), then the physics corresponds, to good approximation and locally in space-time, to that which is measured in an inertial frame.

Possibly, then, the gravitational force is a fictitious force, in analogy with fictitious forces such as the Coriolis force in accelerating systems in Newtonian mechanics. (*Tidal forces*, coming from non-uniform gravitational fields, can in principle be measured and reflect the space-time curvature, as we'll see below.) While Einstein dispensed with a preferred set of observers—and in particular the world-lines of the spatial origins in such coordinate systems—he did postulate that freely falling objects move on time-like geodesics. Geodesics are the analogues of straight lines in a curved space, and Einstein asserted that objects that are experiencing no other force except possibly that of gravity should move along time-like geodesics, while light should propagate along null geodesics.

Whether one tries to ascribe the bending of light discussed earlier to an accelerating frame, or to geometry, if the propagation is along null geodesics, then in any coordinate system, one can compute the covariant acceleration and check that it vanishes. In this way, the paths of light rays in vacuum obey a rule that can be stated invariantly, and thus takes the same form in *every* frame of reference. (This is also true in Minkowski space, of course.) Laws of physics should in principle be able to be formulated in a coordinate-independent way: this is the *Principle of General Covariance.* Tensorial equations have this property: the components of tensors and their covariant derivatives will generally be different in different frames, but the tensors themselves are invariant objects, like geometric quantities. In any coordinate system one can check to see if a curve is a geodesic, not by checking if the path is linear with respect to the coordinates, but rather by computing the covariant acceleration.

Einstein made a remarkable argument that the space-time continuum in the presence of a gravitational field should be non-Euclidean. Consider a Euclidean disk of radius $r$, with a boundary curve $C_r$. Measured in the rest frame of the disk, the length of the circle is $2\pi r$. Consider now measurements made from the point of view of an observer $\mathcal{O}$ moving *along* the circle $C_r$ with angular velocity (relative to the rest frame) $\omega$. Relative to the frame of the disk, the length of a unit measuring stick in the frame $\mathcal{O}$ oriented *tangentially* along $C_r$ is shorter than one unit, by the factor $\sqrt{1-(\omega r/c)^2}$. Thus, the length $L(C_r)$ measured in $\mathcal{O}$ is *more* than $2\pi r$: if $r\omega/c$ is small, then in $\mathcal{O}$, $L(C_r) \approx 2\pi r(1+\frac{1}{2}(\frac{\omega r}{c})^2)$. Another way to think of this is that the observer $\mathcal{O}$ places lots of measuring sticks along the circle, and an observer at rest with respect to the disk counts how many of these are needed to go around the perimeter to determine the length of the circle as measured in $\mathcal{O}$ (remember that the observers will not necessarily agree on simultaneity). On the other hand, distances perpendicular to the direction of motion agree in both frames. Thus both frames agree on the radius $r$. $\mathcal{O}$ concludes that the spatial geometry is *curved*. Indeed, recall the classical formula for the Gauss curvature, which applied to the above analysis would yield non-zero curvature:

$$K(p) = \lim_{r\to 0} \frac{3}{\pi} \cdot \frac{2\pi r - L(C_r)}{r^3}.$$

The geometry is either Euclidean, or it's not. Depending on which it is, one (or both!) of the two frames would need to convert coordinate distances to the truly invariant geometric distances.

The effect of gravity on light has profound implications for the role of gravity in physics. For, since it is currently assumed (based on data) that signals cannot travel faster than the speed of light, then since gravity affects the paths of light rays, gravity then plays a distinguished role in determining the causal structure of the universe. Indeed, consider space-time modeled on a Lorentzian manifold, and assume that light rays move along null geodesics. The collection of null cones determines the conformal class of the space-time metric $g$. Indeed, if $X$ is time-like and $Y$ is space-like at a point $p$, then $g(X + aY, X + aY) = a^2 g(Y,Y) + 2ag(X,Y) + g(X,X)$. The leading coefficient has positive sign, and the constant term is negative, so there are exactly two real roots of this polynomial, which in principle we can glean from knowing the null cone at $p$. The product of these roots gives $\frac{g(X,X)}{g(Y,Y)}$. If $V$ and $W$ are any tangent vectors at $p$, then $g(V,W) = \frac{1}{2}(g(V+W,V+W) - g(V,V) - g(W,W))$. Knowing the light cone at $p$, then, allows us to determine the ratio $\frac{g(V,W)}{g(X,X)}$, since any of the terms on the right of the preceding equation, if non-zero, can be gleaned in ratio with either $g(X,X)$ or $g(Y,Y)$.

## 2.3. The Einstein Equation

Our guiding principles in deriving the Einstein equation are as follows. We start with a quote from Einstein: "The laws of physics must be of such a nature that they apply to systems of reference in any kind of motion." This *Principle of General Relativity* puts all frames of reference on an equal footing, in contrast to the privileged inertial frames of reference of special relativity. We have seen above that global inertial frames appear to be incompatible with gravitation. Furthermore, as we have seen from the *Principle of Equivalence* that accelerating frames ought to be locally equivalent to a uniform gravitational field, so that any coordinate system should be equivalent in a theory incorporating gravity. The *Principle of General Covariance* asserts that it should be possible to formulate the equations of physical laws in a coordinate-independent manner, such as laws that are tensorial in nature, so that the laws of physics have the same form for all frames of reference. Certainly, equations of physical laws in Minkowski space-time can also be put into covariant form, but we will have other space-time metrics when there is a non-trivial gravitational field, which we emphasize can be the case *even in vacuum*, possibly corresponding the the gravitational field outside a compact massive object, say. One seeks to relate the space-time geometry to the distribution of matter fields and energy within space-time: the result of this is the Einstein equation, which will be discussed at length below. From the earlier discussion of the Principle of Equivalence, we assert that space-time should be Lorentzian: thus in normal coordinates at some point $p$ in space-time, the laws of physics will take the same form as they would in special relativity, and thus local to $p$, the laws are approximately of the same form as special relativity. Said another way, in a small region of space-time, local experiments cannot detect a gravitational field, which is roughly uniform, and could be ascribed to a uniformly accelerating frame of reference. In larger regions of space-time, non-uniformities can be detected by measuring tidal forces. As a final guiding principle, the equations governing gravity should yield Newton's Law of Gravitation in the case that the gravitational field is sufficiently "weak."

Einstein searched for a way to relate the geometry of space-time to the energy-momentum distribution of matter and fields it within it. The gravitational field *per se* is encoded in the metric of space-time. Einstein sought to equate the stress-energy tensor $T$ describing the energy-momentum densities of the fields and matter to some tensor created from the metric $g$. As such, he faced some restrictions. Indeed the tensor $T$ was symmetric and divergence-free in special relativity, and so it should be in a general space-time; indeed the identities should persist in normal coordinates (locally inertial frame) about any point $p$, and thus the divergence of $T$ will vanish in any coordinate system, in accordance with general covariance. Einstein originally tried to equate $T$ with the Ricci tensor (up to a scalar multiple), but that was doomed to fail in general, by the contracted Bianchi identity (Corollary 2.3). This is standard, but it is so important we give the proof here.

**Proposition 2.2** (Bianchi Identities). *In a pseudo-Riemannian manifold $(M, g)$, the curvature tensor satisfies an algebraic Bianchi identity,*

$$R(X,Y,Z) + R(Y,Z,X) + R(Z,X,Y) = 0$$

*for all vectors $X$, $Y$, and $Z$. The curvature tensor also satisfies a differential Bianchi identity: for all vectors $X$, $Y$, $Z$, $V$, and $W$,*

$$\langle(\nabla_X R)(V,W,Y),Z\rangle + \langle(\nabla_Y R)(V,W,Z),X\rangle + \langle(\nabla_Z R)(V,W,X),Y\rangle = 0.$$

*By symmetry-by-pairs, this is equivalent to*

$$\langle(\nabla_X R)(V,W,Y),Z\rangle + \langle(\nabla_V R)(W,X,Y),Z\rangle + \langle(\nabla_W R)(X,V,Y),Z\rangle = 0.$$

**Proof.** Recall our convention $R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. Thus we can re-arrange terms to obtain

$$\begin{aligned}
-\Big[R(X,Y,Z) &+ R(Y,Z,X) + R(Z,X,Y)\Big] \\
&= \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y \\
&\quad - \nabla_X(\nabla_Y Z - \nabla_Z Y) - \nabla_Y(\nabla_Z X - \nabla_X Z) - \nabla_Z(\nabla_X Y - \nabla_Y X) \\
&= \nabla_{[X,Y]}Z + \nabla_{[Y,Z]}X + \nabla_{[Z,X]}Y - \nabla_X[Y,Z] - \nabla_Y[Z,X] - \nabla_Z[X,Y] \\
&= [[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] \\
&= 0.
\end{aligned}$$

The last line is the Jacobi identity, and in the preceding line we used the torsion-free property of the Levi-Civita connection: $\nabla_V W - \nabla_W V = [V,W]$.

For the proof of the differential Bianchi identity, we use a coordinate frame $\frac{\partial}{\partial x^i}$. In fact, we use normal coordinates at a point $p \in M$, so that $g_{ij}(p) = \pm\delta_{ij}$, and $\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\Big|_p = 0$. A semicolon is used to denote components of the covariant derivative $\nabla R$. Since $[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}] = 0$, we have at the point $p$:

$$\begin{aligned}
R_{ijk\ell;m}(p) &= \frac{\partial}{\partial x^m}\Big|_p \Big\langle R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\Big), \frac{\partial}{\partial x^\ell}\Big\rangle \\
&= \Big\langle \nabla_{\frac{\partial}{\partial x^m}}\Big(\nabla_{\frac{\partial}{\partial x^i}}\nabla_{\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}}\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^k}\Big), \frac{\partial}{\partial x^\ell}\Big\rangle\Big|_p.
\end{aligned}$$

Thus we have (combining terms in pairs, and using the fact that for each $i$ and $j$, $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}\Big|_p = 0$)

$$
\begin{aligned}
&R_{ijk\ell;m}(p) + R_{jmk\ell;i}(p) + R_{mik\ell;j}(p) = \\
&\quad \left\langle \nabla_{\frac{\partial}{\partial x^m}} \Big( \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} \Big), \frac{\partial}{\partial x^\ell} \right\rangle \Big|_p \\
&\quad + \left\langle \nabla_{\frac{\partial}{\partial x^i}} \Big( \nabla_{\frac{\partial}{\partial x^j}} \nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^m}} \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k} \Big), \frac{\partial}{\partial x^\ell} \right\rangle \Big|_p \\
&\quad + \left\langle \nabla_{\frac{\partial}{\partial x^j}} \Big( \nabla_{\frac{\partial}{\partial x^m}} \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k} - \nabla_{\frac{\partial}{\partial x^i}} \nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^k} \Big), \frac{\partial}{\partial x^\ell} \right\rangle \Big|_p \\
&\quad = \left\langle R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^m}, \nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^k}) + R(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^m}, \nabla_{\frac{\partial}{\partial x^j}} \frac{\partial}{\partial x^k}) \right. \\
&\quad\quad \left. + R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}, \nabla_{\frac{\partial}{\partial x^m}} \frac{\partial}{\partial x^k}), \frac{\partial}{\partial x^\ell}, \right\rangle \Big|_p = 0.
\end{aligned}
$$

□

**Corollary 2.3.** *If $(M, g)$ is a pseudo-Riemannian manifold with scalar curvature $R(g)$, then*

$$2 \operatorname{div}_g(\operatorname{Ric}(g)) = dR(g).$$

**Proof.** We use the differential Bianchi identity below, along with symmetries of the curvature tensor, and the fact that $\nabla g = 0$, so that $g^{ij}_{;k} = 0$ for all $i, j, k$:

$$
\begin{aligned}
dR(g)_i &= (g^{j\ell} g^{km} R_{kj\ell m})_{;i} \\
&= g^{j\ell} g^{km}(-R_{kjmi;\ell} - R_{kji\ell;m}) \\
&= g^{j\ell} g^{km}(R_{jkmi;\ell} + R_{jki\ell;m}) \\
&= g^{j\ell} R_{ji;\ell} + g^{km} R_{ki;m} \\
&= 2(\operatorname{div}_g(\operatorname{Ric}(g)))_i.
\end{aligned}
$$

□

From this lemma, we construct the *Einstein tensor*

$$G_\Lambda = \operatorname{Ric}(g) - \frac{1}{2} R(g)\, g + \Lambda g,$$

where the constant $\Lambda$ is called the *cosmological constant*. We recall that our signature for Lorentzian metrics is $(-, +, +, \ldots, +)$. The Einstein tensor is divergence free, as is any constant scalar multiple, and thus provides a candidate for the stress-energy tensor of space-time. In fact, it is known that up to scalar multiple, $G_\Lambda$ is the *only* divergence-free symmetric tensor whose coordinate expression is a function of the components $(g_{\mu\nu})$ of the metric tensor, along with their first and second partial derivatives. This result was known to Cartan and Weyl in the special case that the tensor is quasi-linear, and the more general result was proved by Lovelock.

From this result, then if the Einstein equation should be as simple as possible, and thus be second-order in the metric components, then it must take the form

$$G_\Lambda = \kappa T$$

for some constant $\kappa$ that will be determined by the Newtonian limit, as we now show.

**2.3.1. The Newtonian Limit.** We first consider a space-time metric which is close to the Minkowski metric, so that in coordinates $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is taken to be "small." Our coordinates $x^\mu$ are chosen so that $x^0 = ct$, and we assume that $g_{\mu\nu,0} = 0$. Consider a geodesic parametrized by proper time $\tau$. We consider the geodesic to model a slowly moving particle trajectory, so that $\left|\frac{dx^i}{d\tau}\right| \ll c\frac{dt}{d\tau} \approx c$. We will expand to first order in $h$ (and derivatives of $h$) and $\frac{1}{c}\frac{dx^i}{dt}$, and denote expressions that are equal up to terms quadratic in these quantities using "$\sim$." Since $g^{\mu\nu} = \eta^{\mu\nu} + O(h)$, we have

$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu\nu}(g_{\nu 0,0} + g_{0\nu,0} - g_{00,\nu}) \sim -\frac{1}{2}\eta^{\mu\nu}h_{00,\nu}.$$

Since $\Gamma^\mu_{\rho\sigma} = O(h)$, the geodesic equation becomes

$$0 = \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \sim \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dx^0}{d\tau}\right)^2 \sim \frac{d^2x^\mu}{d\tau^2} + c^2\Gamma^\mu_{00}.$$

Consider the spatial components:

$$\frac{d^2x^i}{d\tau^2} \sim -c^2\Gamma^i_{00} = \frac{1}{2}c^2h_{00,i}.$$

If we let $\Phi = -\frac{1}{2}c^2h_{00}$, so that $g_{00} = -(1+\frac{1}{2}\Phi c^{-2})$, then we recover the Newtonian relation between acceleration and the gradient of the gravitational potential (2.1).

We still want to determine $\kappa$. To do this, let's consider the stress-energy for a dust model. For a perfect fluid, the pressure becomes important due to high random motion of the particles, and we are assuming our particles are slowly moving. So we are just considering the particles at rest in a given frame, without any pressure forces between dust particles, each with four-velocity $\mathbb{U}$. As such $T^{\mu\nu} = c^{-2}\rho\mathbb{U}^\mu\mathbb{U}^\nu$. We again consider $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $g_{\mu\nu,0} = 0$ (or at least $g_{\mu\nu,0} \sim 0$). We expand to first order in $h$ (and its derivatives), $\mathbb{U}^i$ and $\rho$. Since $g_{\mu\nu}\mathbb{U}^\mu\mathbb{U}^\nu = -c^2$, $\mathrm{tr}_g(T) = g_{\mu\nu}T^{\mu\nu} = -\rho$, $g_{00} = -1 + h_{00}$, and we have $\mathbb{U}^0 \sim c + \frac{1}{2}c^2h_{00}$. Furthermore,

$$T_{00} = g_{\mu 0}g_{\nu 0}T^{\mu\nu} \sim g_{00}g_{00}T^{00} = (-1 + h_{00})^2c^{-2}\rho\mathbb{U}^0\mathbb{U}^0 \sim \rho.$$

From the Einstein equation (with $\Lambda = 0$) $\mathrm{Ric}(g) - \frac{1}{2}R(g)g = \kappa T$, taking a trace yields $R(g) = \kappa\rho$, which then yields, upon evaluating the $(0,0)$-component of the Einstein equation, that $R_{00} + \frac{1}{2}\kappa\rho \sim \kappa\rho$, or $R_{00} \sim \frac{1}{2}\kappa\rho$. We can also compute $R_{00}$ in terms of Christoffel symbols. Indeed we have

$$\begin{aligned} R^i_{j00} &= \Gamma^i_{00,j} - \Gamma^i_{j0,0} + \Gamma^i_{j\mu}\Gamma^\mu_{00} - \Gamma^i_{0\mu}\Gamma^\mu_{j0} \\ &\sim \Gamma^i_{00,j} = \left[\frac{1}{2}g^{i\mu}(g_{\mu 0,0} + g_{0\mu,0} - g_{00,\mu}\right]_{,j} \\ &\sim -\frac{1}{2}\delta^{i\mu}g_{00,\mu j} \\ \frac{1}{2}\kappa\rho = R_{00} = R^i_{i00} &\sim -\frac{1}{2}\Delta(h_{00}) = c^{-2}\Delta\Phi \end{aligned}$$

To compare with the Newtonian limit, we convert the energy density to mass density, $\sigma = c^{-2}\rho$, to obtain

$$\Delta\Phi = \frac{1}{2}\kappa c^4\sigma.$$

Thus we have $\frac{1}{2}\kappa c^4 = 4\pi G$, or

$$\kappa = \frac{8\pi G}{c^4}.$$

**2.3.2. Energy conditions.** We note that the Einstein equation is sometimes re-written as follows. From $\mathrm{Ric}(g) - \frac{1}{2}R(g)g + \Lambda g = \kappa T$, we trace to obtain (in space-time dimension four) $-R(g) + 4\Lambda = \kappa\,\mathrm{tr}_g(T)$, so

$$\mathrm{Ric}(g) = \kappa\Big(T - \frac{1}{2}\mathrm{tr}_g(T)g\Big) + \Lambda g. \tag{2.3}$$

In the vacuum case $T = 0$, so the Einstein equation becomes $\mathrm{Ric}(g) = \Lambda g$. One could take $-\Lambda g$ to part of $T$, and thus the vacuum Einstein equation commonly refers to $\mathrm{Ric}(g) = 0$.

Before we move on, we note that conditions coming from physical notions are sometimes imposed on $T$. We mention several now. The *weak energy condition* is that $T(\mathbb{U},\mathbb{U}) \geq 0$ for all time-like $\mathbb{U}$. If $c = 1$, say, then unit time-like vectors $\mathbb{U}$ correspond to physical observers, and $T(\mathbb{U},\mathbb{U})$ is the energy density as measured by such an observer. The *strong energy condition* is that for all unit time-like $\mathbb{U}$, $T(\mathbb{U},\mathbb{U}) \geq -\frac{1}{2}\mathrm{tr}_g(T)$. From (2.3), we see this is equivalent in case $\Lambda = 0$ to the *time-like convergence condition* $\mathrm{Ric}(\xi,\xi) \geq 0$ for all time-like $\xi$. In a time-oriented Lorentz manifold (i.e. if the manifold admits a smooth time-like vector field that can be used to give a smooth assignment of a future time-cone in the tangent space at each point), we define the *dominant energy condition* that for all future-directed time-like $\xi$, the vector given by $-T^a{}_b\xi^b$ is future-directed causal, or in other words, for all future-directed time-like (causal) $\xi$ and $\chi$, we have $T(\xi,\chi) \geq 0$. The dominant energy condition clearly implies the weak energy condition.

**2.3.3. The Einstein Equation in Fermi Coordinates along a Time-like Geodesic.** Consider again motion of particles under gravitational force with potential $\Phi$ in the Newtonian framework. We consider a family of paths $\xi(t,s)$ with coordinates $x^k(t,s)$, where $s$ parametrizes the family of paths by, say, their initial position $s$ along an axis. Newton's law becomes $\frac{\partial^2 x^k}{\partial t^2} = -\frac{\partial \Phi}{\partial x^k}(\xi(t,s))$. We now consider the variation vector $V(t) = \frac{\partial \xi}{\partial s} = \frac{\partial x^k}{\partial s}\frac{\partial}{\partial x^k}$ in the direction across nearby paths. This vector satisfies the equation

$$\left(\frac{D^2 V}{dt^2}\right)^k = \frac{\partial^2}{\partial t^2}\left(\frac{\partial x^k}{\partial s}\right) = -\left(\frac{\partial^2 \Phi}{\partial x^j \partial x^k}\right)\frac{\partial x^j}{\partial s} = -\left(\frac{\partial^2 \Phi}{\partial x^j \partial x^k}\right)V^j(s). \tag{2.4}$$

This equation describes the relative motion of particles moving on nearby paths under the force of gravity. The relative motion is sometimes described in terms of *tidal forces* from non-uniformities in the gravitational field, which is consistent with what we see in the equation. The matrix which governs this behavior is negative the Hessian of $\Phi$, so the negative of its trace is $\Delta\Phi = 4\pi G\sigma$, where $\sigma$ is the mass density. In the vacuum case $\Delta\Phi = 0$, and this means that some nearby paths accelerate toward each other, while others accelerate away from each other.

We can use this formulation as a motivation for Einstein's equation. In general relativity, the paths of observers subject only to gravitation ("freely falling" observers) are given by time-like geodesics. We recall the logic behind this. By the equivalence principle, a gravitational field can be created or effectively cancelled out locally in space-time by a coordinate change. Indeed, if $\gamma(t)$ is a geodesic, then the geodesic equation in coordinates is $\frac{d^2\gamma^k}{dt^2} = -\Gamma^k_{ij}(\gamma(t))\frac{d\gamma^i}{dt}\frac{\gamma^j}{dt}$. In normal coordinates at $p = \gamma(0)$, these equations reduce to $\left.\frac{d^2\gamma^k}{dt^2}\right|_0 = 0$, analogous to the Newtonian equation with vanishing gravitational field. The Christoffel symbols do not form the components of a tensor field, and can be transformed away at a point by a coordinate change. Normal coordinates at a point in space-time form a *momentarily co-moving rest frame*, or locally inertial frame, associated to the observer. In normal coordinates at such a point, covariant derivatives reduce to partial derivatives, and tensor equations take the same form as they would have in inertial coordinates in Minkowski space (special relativity).

We now derive the equation governing the behavior of a family of nearby geodesics. Let $f(t,s)$ be a two-parameter map, so that for each $s$, $\gamma_s(t) = f(t,s)$ is a geodesic. The vector field $V = \frac{\partial f}{\partial s}$ along $f$ is the *variation field* for the family of geodesics. If we focus on $\gamma = \gamma_0$ and consider $V(t)$ along $\gamma$, then $V(t)$ satisfies the following differential equation, the *Jacobi equation.*

**Proposition 2.4.** *Consider a family of geodesics $f(t,s)$ as above, with variation field $V$. Then along $\gamma = f(\cdot, 0)$,*

$$\frac{D^2V}{dt^2} = R(\gamma'(t), V(t), \gamma'(t)).$$

**Proof.** We first note the following:

$$\begin{aligned}
\frac{D}{dt}\frac{\partial f}{\partial s} &= \nabla_{\gamma'(t)}\frac{\partial f}{\partial s} \\
&= \nabla_{\gamma'(t)}\left(\frac{\partial f^k}{\partial s}\frac{\partial}{\partial x^k}\right) \\
&= \frac{\partial^2 f^k}{\partial t \partial s}\frac{\partial}{\partial x^k} + \frac{\partial^2 f^k}{\partial t \partial s}\frac{d\gamma^\ell}{dt}\nabla_{\frac{\partial}{\partial x^\ell}}\frac{\partial}{\partial x^k} \\
&= \left(\frac{\partial^2 f^m}{\partial t \partial s} + \frac{\partial f^k}{\partial s}\frac{df^\ell}{dt}\Gamma^m_{k\ell}\right)\frac{\partial}{\partial x^m} \\
&= \frac{D}{ds}\frac{\partial f}{\partial t}.
\end{aligned}$$

Now, from the definition of curvature ($R(X,Y,Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$), and since $0 = df\left(\left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right]\right) = \left[\frac{\partial f}{\partial s}, \frac{\partial f}{\partial t}\right]$, we have (with $\frac{D}{dt} = \nabla_{\gamma'(t)}$ and $\frac{D}{ds} = \nabla_{\frac{\partial f}{\partial s}} = \nabla_{V(t)}$)

$$\begin{aligned}
\frac{D^2V}{dt^2} &= \frac{D}{dt}\frac{D}{dt}\frac{\partial f}{\partial s} = \frac{D}{dt}\frac{D}{ds}\frac{\partial f}{\partial t} \\
&= \frac{D}{ds}\frac{D}{dt}\frac{\partial f}{\partial t} + R(\gamma'(t), V(t), \gamma'(t)) = R(\gamma'(t), V(t), \gamma'(t)),
\end{aligned}$$

since $\frac{D}{dt}\frac{\partial f}{\partial t} = \nabla_{\gamma'(t)}\gamma'(t) = 0$ since $\gamma$ is a geodesic. □

If we compare this to (2.4), we see that the analogue of the matrix $\left(\frac{\partial^2\Phi}{\partial x^j \partial x^k}\right)$ is the matrix $\left(R^k_{j\ell m}\frac{d\gamma^\ell}{dt}\frac{d\gamma^m}{dt}\right)$, so that the analogue of $\Delta\Phi$ is obtained by tracing over $j$ and $k$ to obtain $\mathrm{Ric}(\gamma'(t), \gamma'(t))$, where $t$ is proper time.

So we see that the analogue of the Newtonian law of gravitation with vanishing field is $\mathrm{Ric}(g) = 0$, since $\mathrm{Ric}(\gamma'(t), \gamma'(t)) = 0$ for all time-like directions at a point implies the Ricci curvature must vanish.

To make a further link to Newtonian theory, we now construct coordinates along a time-like geodesic adapted to the geodesic and the geometry. Consider a time-like geodesic $\gamma(\tau)$, parametrized by proper time $\tau$, $\gamma(0) = p$. Choose an orthonormal frame $\{e_1, e_2, e_3\}$ for the orthogonal complement of $\gamma'(0)$ in $T_pM$. We parallel translate these vectors along $\gamma$ to produce an orthonormal frame $\{e_1(\tau), e_2(\tau), e_3(\tau)\}$ for the orthogonal complement of $\gamma'(t)$ in $T_{\gamma(\tau)}M$. We define coordinates in a neighborhood of the geodesic by the map $\varphi(\tau, x) = \exp_{\gamma(\tau)}(x^i e_i(\tau))$. Since $d\varphi\Big(\frac{\partial}{\partial\tau}\Big|_{(\tau,0)}\Big) = \gamma'(\tau)$, and $d\varphi\Big(\frac{\partial}{\partial x^i}\Big|_{(\tau,0)}\Big) = e_i(\tau)$, we see that $\varphi$ defines a coordinate system in a neighborhood of $\gamma$, *Fermi coordinates*. For index purposes, let $x^0 = c\tau$.

It is clear by construction that the metric components along $\gamma$ in this coordinate system agree with the components of the Minkowski metric in inertial coordinates. We now establish a lemma regarding the behavior of the Christoffel symbols along $\gamma$.

**Lemma 2.5.** *For all* $0 \le i, j, k \le 3$, $\Gamma^k_{ij}(c\tau, 0) = \Gamma^k_{ij}(\gamma(\tau)) = 0$.

**Proof.** For any $(\tau, b)$, $b = (b^1, b^2, b^3)$, consider the curve defined via the exponential map as $\beta(s) = \exp_{\gamma(\tau)}(sb^i e_i(\tau))$. Let $b^0 = 0$. Then $\beta^0(s) = c\tau$ and $\beta^k(s) = sb^k$ for $k = 1, 2, 3$. By definition, $\beta$ is a geodesic. Since $\frac{d^2\beta^k}{ds^2} = 0$ for $k = 0, 1, 2, 3$, we have $0 = \Gamma^k_{ij}(c\tau, sb)\frac{d\beta^i}{ds}\frac{d\beta^j}{ds} = \Gamma^k_{ij}(c\tau, sb)b^i b^j$. Since at $s = 0$, we can consider $\beta$ defined by an arbitrary $b \in \mathbb{R}^3$, we have $\Gamma^k_{ij}(c\tau, 0) = 0$ for $0 \le k \le 3$ and $1 \le i, j \le 3$. Similarly, the geodesic equation for $\gamma$ yields $\Gamma^k_{00}(c\tau, 0) = 0$ for $0 \le k \le 3$.

To get the other Christoffel symbols, we use the equations for parallel transport. Along $\gamma$, the $x^\ell$-coordinate vector is $e_\ell$ for $\ell = 1, 2, 3$, so the components of $e_\ell$ along $\gamma$ are given by $e^i_\ell(\tau) = \delta^i{}_\ell$. The parallel transport equations are then given by

$$0 = \frac{d}{d\tau}(e^k_\ell(\tau)) + \Gamma^k_{ij}(c\tau, 0)e^i_\ell(\tau)\frac{d\gamma^j}{d\tau} = \Gamma^k_{ij}(c\tau, 0)\delta^i{}_\ell\frac{d\gamma^j}{d\tau} = c\Gamma^k_{\ell 0}(c\tau, 0),$$

where we used the fact that $\gamma^0(\tau) = c\tau$ and $\gamma^j(\tau) = 0$ for $j = 1, 2, 3$. □

**Lemma 2.6.** *Along* $\gamma$, *we have* $R^k_{j00} = \frac{\partial\Gamma^k_{00}}{\partial x^j}$, *so that for* $k = 1, 2, 3$, $R^k_{j00} = -\frac{1}{2}g_{00,kj}$, *while* $R^0_{j00} = 0$, *along* $\gamma$.

**Proof.** Since the Christoffel symbols vanish along $\gamma$, $\frac{\partial\Gamma^k_{ij}}{\partial\tau} = 0$ along $\gamma$, so that along $\gamma$,

$$\begin{aligned} R^k_{j00}\frac{\partial}{\partial x^k} &= \nabla_{e_j(\tau)}\nabla_{\frac{\partial}{\partial x^0}}\frac{\partial}{\partial x^0} - \nabla_{c^{-1}\gamma'(\tau)}\nabla_{\frac{\partial}{\partial x^k}}\frac{\partial}{\partial x^0} \\ &= \nabla_{e_j(\tau)}\Big(\Gamma^k_{00}\frac{\partial}{\partial x^k}\Big) - c^{-1}\nabla_{\gamma'(\tau)}\Big(\Gamma^k_{j0}\frac{\partial}{\partial x^k}\Big) \\ &= \frac{\partial\Gamma^k_{00}}{\partial x^j}\frac{\partial}{\partial x^k}. \end{aligned}$$

Moreover, since the Christoffel symbols vanish along $\gamma$, so do the first partials of $g_{ij}$ and $g^{ij}$. Thus along $\gamma$, we have for $k = 1, 2, 3$,

$$\frac{\partial \Gamma^k_{00}}{\partial x^j} = \frac{\partial}{\partial x^j}\left(\frac{1}{2}g^{km}(2g_{0m,0} - g_{00,m})\right) = -\frac{1}{2}\delta^{km}g_{00,mj} = -\frac{1}{2}g_{00,kj}.$$

Similarly, $\frac{\partial \Gamma^0_{00}}{\partial x^j} = -\frac{1}{2}g_{00,0j} = 0$. □

Now, as we noted above, the analogue of the matrix $\left(\frac{\partial^2 \Phi}{\partial x^j \partial x^k}\right)$ is

$$c^2 R^k_{j00} = -\frac{1}{2}c^2 g_{00,jk}.$$

Thus the analogue of the gravitational potential $\Phi$ is $-\frac{1}{2}c^2 g_{00}$, which is equivalent to what we had earlier (since $\Phi$ is defined only up to an additive constant). The analogue of $\Delta\Phi$ is then $\mathrm{Ric}(\gamma'(\tau), \gamma'(\tau)) = c^2 R_{00}$, as we had before. Our analysis leads us again to propose that the Ricci curvature should be related to the matter density. Of course, the mass-energy density is not an invariant object, but the stress-energy tensor is. We can argue as in our earlier analysis how to get from here to the Einstein equation.

**2.3.4. Variational formulation.** We now consider a Lagrangian variational formulation for the Einstein equation, as first derived by Hilbert about a hundred years ago. Consider the *total scalar curvature functional* $\mathcal{R}(g) = \int_M R(g)dv_g$, where in local coordinates, $dv_g = \sqrt{|\det(g_{ij})|}\,dx$ (the definition then includes both Lorentzian and Riemannian cases). We assume that $M$ is compact, or more generally that $R(g) \in L^1(M, dv_g)$. We want to compute the first variation of $\mathcal{R}$, and the associated Euler-Lagrange equation.

First we recall *Cramer's Rule*. If $A = (A_{ij})$ is an $n \times n$ matrix, then we let $M_{ij}$ be the determinant of the $(n-1)\times(n-1)$ *minor* matrix obtained by deleting row $i$ and column $j$ of $A$. The determinant of $A$ is given by column or row expansion:

$$\det(A) = \sum_{i=1}^{n}(-1)^{i+j}M_{ij}A_{ij} = \sum_{j=1}^{n}(-1)^{i+j}M_{ij}A_{ij}.$$

We thus let $\mathrm{cof}(A)$ be the $n\times n$ *cofactor matrix*, $\mathrm{cof}(A)_{ij} = (-1)^{i+j}M_{ij}$, and we let $A^{\mathrm{adj}} = (\mathrm{cof}(A))^T$ be the *Cramer's Rule adjoint* with entries $A^{\mathrm{adj}}_{ij} = (-1)^{i+j}M_{ji} = \mathrm{cof}(A)_{ji}$. Thus for any $j \in \{1, 2, \ldots, n\}$, $\det(A) = (A^{\mathrm{adj}}\cdot A)_{jj}$. We also note that for $i \neq j$, $(A^{\mathrm{adj}}\cdot A)_{ij} = \sum_{k=1}^{n}(-1)^{i+k}M_{ki}A_{kj} = 0$. Indeed, we can interpret the preceding sum as the determinant of the matrix $\tilde{A}$ obtained by replacing column $i$ of $A$ by column $j$ of $A$. The minors $M_{ki}$ are obtained by crossing out column $i$, so $M_{ki}$ are the same for $A$ as for $\tilde{A}$. On the other hand, $\det(\tilde{A}) = 0$ since $\tilde{A}$ has two equal columns. In summary we arrive at *Cramer's Rule*: If $I_n$ is the $n \times n$ identity matrix, then $A^{\text{adj}} \cdot A = \det(A) I_n$. If $A$ is invertible, then

$$A^{-1} = \frac{1}{\det(A)} A^{\text{adj}}.$$

Now we turn to variational formulae.

**Lemma 2.7.** *If $A(t)$ is a smooth path of $n \times n$ matrices,*

$$\frac{d}{dt}\left(\det(A(t))\right) = \operatorname{tr}(A^{\text{adj}}(t) \cdot A'(t)). \tag{2.5}$$

*In case $A(t)$ is invertible,*

$$\frac{d}{dt}\left(\log(\det(A(t)))\right) = \operatorname{tr}(A^{-1}(t) \cdot A'(t)), \tag{2.6}$$

*and*

$$\frac{d}{dt}\sqrt{|\det A(t)|} = \frac{1}{2}\sqrt{|\det(A(t))|}\,\operatorname{tr}(A^{-1}(t) \cdot A'(t)). \tag{2.7}$$

**Proof.** If consider $\det(A)$ as a function of $(A_{ij}) \in \mathbb{R}^{n^2}$, then

$$\frac{\partial}{\partial A_{ij}}(\det A) = (-1)^{i+j} M_{ij} = A^{\text{adj}}_{ji}.$$

Thus by the Chain Rule, $\frac{d}{dt}\left(\det(A(t))\right) = \sum_{i,j=1}^{n} \frac{\partial(\det(A))}{\partial A_{ij}} \frac{\partial A_{ij}}{\partial t}$. Using this together with the preceding equation establishes the lemma. □

We need one more variational formula to compute the Euler-Lagrange equation for the Einstein-Hilbert action. The proof can be carried out in a straightforward though laborious manner, in normal coordinates at a point. We leave it as an exercise. Let $L_g(h) = \frac{d}{dt}\Big|_{t=0} R(g + th)$.

**Lemma 2.8.** $L_g(h) = -\Delta_g(\operatorname{tr}_g(h)) + \operatorname{div}_g \operatorname{div}_g h - h \cdot \operatorname{Ric}(g)$.

**Exercise 2.9.** Prove Lemma 2.8.

We now derive the Euler-Lagrange equation for the Einstein-Hilbert action. We will vary the metric $g$ in the direction of a symmetric $(0, 2)$-tensor $h$. We will take $h$ to be compactly supported, so we can make sense out of the variation for a given $h$ even in case $R(g)$ fails to be integrable, by integrating only over the support of $h$, where the metric $g$ is actually changing.

**Theorem 2.10.** *The first variation of the Einstein-Hilbert action (total scalar curvature functional) $\mathcal{R}$ is given by*

$$\frac{d}{dt}\Big|_{t=0}\mathcal{R}(g+th) = -\int_M h\cdot(\mathrm{Ric}(g) - \frac{1}{2}R(g)g)\,dv_g$$

*for all compactly supported tensors $h$ (which vanish on the boundary $\partial M$ if $\partial M$ is nonempty). Thus the Euler-Lagrange equation is* $\mathrm{Ric}(g) - \frac{1}{2}R(g)g = 0$*. This equation is satisfied on all two-dimensional manifolds $(M,g)$. For* $\dim(M)\geq 3$*, the Euler-Lagrange equation is equivalent to $Ric(g) = 0$.*

**Proof.** Since in local coordinates $x$, $dv_g = \sqrt{|\det(g_{ij})|}\,dx$, we have by the preceding lemma (and the symmetry of $g$ (or $h$)) and the fact that $g^{ij}h_{ij} = \mathrm{tr}_g(h)$,

$$\frac{d}{dt}\Big|_{t=0}dv_g = \frac{1}{2}g^{ij}h_{ij}dv_g = \frac{1}{2}\mathrm{tr}_g(h)\,dv_g. \tag{2.8}$$

Thus we have by integration by parts (boundary terms vanish by choice of $h$)

$$\begin{aligned}\frac{d}{dt}\Big|_{t=0}\mathcal{R}(g+th) &= \int_M L_g(h)\,dv_g + \int_M R(g)\cdot\frac{1}{2}\mathrm{tr}_g(h)\,dv_g \\ &= \int_M \left[(-\Delta_g(\mathrm{tr}_g(h)) + \mathrm{div}_g\mathrm{div}_g h - h\cdot\mathrm{Ric}(g)) + \frac{1}{2}R(g)g\cdot h\right]\,dv_g \\ &= -\int_M h\cdot(\mathrm{Ric}(g) - \frac{1}{2}R(g)g)\,dv_g.\end{aligned}$$

In case $M$ is closed, we let $h = \mathrm{Ric}(g) - \frac{1}{2}R(g)g$ to finish the proof. In any case, the preceding equation holds for all $h$ in a dense subset of $L^2(M, dv_g)$, so we see that we must have $\mathrm{Ric}(g) - \frac{1}{2}R(g)g = 0$.

If $M$ is two-dimensional, and $\{e_1, e_2\}$ is an orthonormal basis of $T_pM$, say $\langle e_1, e_2\rangle = 0$, $\langle e_1, e_1\rangle = \epsilon_1 = \pm 1$, and $\langle e_2, e_2\rangle = \epsilon_2 = 1$, then

$$\begin{aligned}\mathrm{Ric}(e_1, e_j) &= \epsilon_1\langle R(e_1, e_1, e_j), e_1\rangle + \epsilon_2\langle R(e_2, e_1, e_j), e_2\rangle \\ &= \langle R(e_2, e_1, e_1), e_2\rangle\delta_{1j} \\ &= \epsilon_1\langle R(e_2, e_1, e_1), e_2\rangle g(e_1, e_j) \\ \mathrm{Ric}(e_2, e_j) &= \epsilon_1\langle R(e_2, e_1, e_1), e_2\rangle\delta_{2j} \\ &= \epsilon_1\langle R(e_2, e_1, e_1), e_2\rangle g(e_2, e_j).\end{aligned}$$

Hence the desired equation follows from

$$R(g) = \epsilon_1\mathrm{Ric}(e_1, e_1) + \epsilon_2\mathrm{Ric}(e_2, e_2) = 2\epsilon_1\langle R(e_2, e_1, e_1), e_2\rangle.$$

If $n = \dim(M) \geq 3$, we trace the Euler-Lagrange equation to obtain $R(g) - \frac{n}{2}R(g) = 0$, or $R(g) = 0$. Thus $\mathrm{Ric}(g) = 0$ in this case; the converse is trivial. □

**Definition 2.11.** A semi-Riemannian manifold $(M^n, g)$ is *Einstein* provided for some constant $C$, $\mathrm{Ric}(g) = Cg$. In this case the scalar curvature is constant: $R(g) = nC$.

**Lemma 2.12.** *Consider $(M^n, g)$ semi-Riemannian with $n \geq 3$. If $\mathrm{Ric}(g) = fg$ for some function $f$, then $f$ is constant.*

**Exercise 2.13.** Prove the lemma.

For any constant $\Lambda$, we can also consider $\mathcal{R}_\Lambda(g) = \int_M (R(g) - 2\Lambda)\, dv_g$. From the variation of the volume element $dv_g$ (2.8), the following is immediate.

**Corollary 2.14.**

$$\frac{d}{dt}\Big|_{t=0} \mathcal{R}_\Lambda(g + th) = -\int_M h \cdot (\mathrm{Ric}(g) - \frac{1}{2}R(g)g + \Lambda g)\, dv_g$$

*for all compactly supported tensors $h$ (which vanish on the boundary $\partial M$ if $\partial M$ is nonempty). Thus the Euler-Lagrange equation for $\mathcal{R}_\Lambda$ is*

$$\mathrm{Ric}(g) - \frac{1}{2}R(g)g + \Lambda g = 0.$$

*Suppose $g$ solves the above equation. Then, for any $\Lambda$ and for $n > 2$, we have $R(g) = \frac{2n}{n-2}\Lambda$. Thus the metric $g$ is* Einstein, $\mathrm{Ric}(g) = Cg$, *where $C$ is the constant $C = \frac{2\Lambda}{n-2}$. In case $n = 2$, $\Lambda = 0$.*

**Definition 2.15.** The *Einstein tensor* is given by $G = G(g) = \mathrm{Ric}(g) - \frac{1}{2}R(g)g$. We also defined the related tensor $G_\Lambda(g) = \mathrm{Ric}(g) - \frac{1}{2}R(g)g + \Lambda g$. The *vacuum Einstein equation* (with cosmological constant $\Lambda$) is given by $G_\Lambda = 0$. In case $\Lambda = 0$ and $n > 2$, this is equivalent to $\mathrm{Ric}(g) = 0$.

We now discuss how to include matter fields into the variational formulation. We assume the matter fields are given in terms of a collection $\Psi$ of tensor fields (including scalars, vectors, etc.), which we assume are *independent* of the metric $g$ and are governed by an action $\int_M \widehat{\mathcal{L}}_m(g, \Psi)\, dv_g$. Note that $\widehat{\mathcal{L}}_m$ can also depend on derivatives of the fields in $\Psi$. If $h$ is a symmetric $(0,2)$-tensor, and $\Phi$ is a collection of (smooth, compactly supported and vanishing on the boundary, if nonempty) tensor fields representing a direction of variation of $\Psi$, we write

$$\frac{d}{dt}\Big|_{t=0} \widehat{\mathcal{L}}_m(g+th,\Phi) =: (D_1\widehat{\mathcal{L}}_m)_{(g,\Psi)}(h,0)$$
$$\frac{d}{dt}\Big|_{t=0} \widehat{\mathcal{L}}_m(g,\Psi+t\Phi) =: (D_2\widehat{\mathcal{L}}_m)_{(g,\Psi)}(0,\Phi).$$

We remark that it is sometimes useful, especially when varying the metric, to express the Lagrangian as a density, $\mathcal{L}_m = \widehat{\mathcal{L}}_m \cdot \sqrt{|\det g|}$.

The fields $\Psi$ are not arbitrary: for any $g$, and for any direction of variation $\Phi$ of the fields, we must have

$$0 = \frac{d}{dt}\Big|_{t=0} \int_M \widehat{\mathcal{L}}_m(g,\Psi+t\Phi)\, dv_g = \int_M (D_2\widehat{\mathcal{L}}_m)_{(g,\Psi)}(0,\Phi)\, dv_g. \tag{2.9}$$

From here we can derive field equations (sometimes called *equations of motion*) for the matter fields. Instead of introducing new notation, we illustrate with examples.

**Example 2.16.** Let's consider $g=\eta$, the Minkowski metric on $\mathbb{M}^{1+k} = \mathbb{R}^{1+k}_1$. Consider global inertial coordinates $(t,x^i)$. Then for any smooth function $\psi$, $\eta(d\psi,d\psi) = -c^{-2}(\frac{\partial\psi}{\partial t})^2 + \sum_{i=1}^{k}(\frac{\partial\psi}{\partial x^i})^2$. If our Lagrangian $\widehat{\mathcal{L}}_m = -\frac{1}{2}\eta(d\psi,d\psi)$, then for any smooth $\varphi$, the field equation is given by

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \int_{\mathbb{M}} -\frac{1}{2}\eta(d\psi+td\varphi, d\psi+td\varphi)\, dt\, dx^1\cdots dx^n \\ &= \int_{\mathbb{M}} -\eta(d\psi,d\varphi)\, dt\, dx^1\cdots dx^k \\ &= \int_{\mathbb{M}} \varphi\Box\psi\, dt\, dx^1\cdots dx^k \end{aligned}$$

where $\Box = -c^{-2}\left(\frac{\partial}{\partial t}\right)^2 + \sum_{i=1}^{k}\left(\frac{\partial}{\partial x^i}\right)^2$ is the *wave operator* in Minkowski space. Since this must hold for all (appropriate) $\varphi$, the field equation is the *wave equation* $\Box\psi = 0$.

**Exercise 2.17.** If we let $\widehat{\mathcal{L}}_m = -\frac{1}{2}\eta(d\psi,d\psi) - V(\psi)$, where $V$ is a smooth function of one variable, show that the resulting field equation is $\Box\psi = V'(\psi)$. In case $V(\psi) = \frac{1}{2}m^2\psi^2$, the resulting equation $\Box\psi - m^2\psi = 0$ is called the *Klein-Gordon equation.*

We remark that the Lagrangian for the gravitational fields involved second-order derivatives of the metric, as does the corresponding Euler-Lagrange equation (the Einstein equation), whereas for the preceding two examples, the Lagrangian is first-order in $\psi$, while the Euler-Lagrange equation is second-order in $\psi$.

**Example 2.18.** For the source-free electromagnetic field, one has $\widehat{\mathcal{L}}_m = -\frac{1}{16\pi} F_{ab}F_{cd}g^{ac}g^{bd}$, where $F_{ab} = (dA)_{ab}$ for a one-form $A$, the *vector potential*. If $\varphi$ is a direction of variation of the vector potential, then the stationarity of the action requires (we use the antisymmetry of $F$)

$$\begin{aligned} 0 &= \int_M -\frac{1}{8\pi} g(d\varphi, F)\, dv_g = \int_M -\frac{1}{8\pi}(\varphi_{b;a} - \varphi_{a;b})F_{cd}g^{ac}g^{bd}\, dv_g \\ &= \int_M \frac{1}{4\pi}\varphi_{a;b}F_{cd}g^{ac}g^{bd}\, dv_g \\ &= \int_M -\frac{1}{4\pi}\varphi_a(F_{cd;b}g^{bd})g^{ac}\, dv_g. \end{aligned}$$

In other words, we obtain the equation $\mathrm{div}_g F = 0$, where $\mathrm{div}_g$ is the *space-time* divergence $F_{cd;b}g^{bd}$, which forms part of Maxwell's equations. The other part comes from the fact that $F_{ab}$ is a closed two-form.

We now define the stress-energy tensor $T$ corresponding to matter fields given by a Lagrangian as above via the following equation identifying $T$ as a symmetric $(0,2)$-tensor via an integral pairing:

$$\begin{aligned} \frac{d}{dt}\Big|_{t=0} \int_M \widehat{\mathcal{L}}_m(g+th, \Psi)\, dv_g &= \int_M \left[(D_1\widehat{\mathcal{L}}_m)_{(g,\Psi)}(h,0) + \widehat{\mathcal{L}}_m(g,\Psi)\cdot \frac{1}{2}\mathrm{tr}_g(h)\right] dv_g \\ &=: \frac{1}{2}\int_M g(h,T)\, dv_g. \end{aligned} \tag{2.10}$$

**Example 2.19.** Consider a Klein-Gordon field $\widehat{\mathcal{L}}_m = -\frac{1}{2}\eta(d\psi, d\psi) - \frac{1}{2}m^2\psi^2$ at the Minkowski metric on $\mathbb{M}^{1+k}$. In inertial coordinates, $\psi_{;a} = \frac{\partial \psi}{\partial x^a}$, $a = 0, 1, \ldots, k$. The stress tensor $T$ satisfies

$$\int_{\mathbb{M}} \eta(h,T)\, dv_\eta = 2\int_{\mathbb{M}} [\frac{1}{2}\eta^{ij}h_{jk}\eta^{k\ell}\psi_{;i}\psi_{;\ell} - (\frac{1}{2}\eta(d\psi, d\psi) + \frac{1}{2}m^2\psi^2)\frac{1}{2}h_{jk}\eta^{jk}\, dv_\eta.$$

Thus we see $T_{ab} = \psi_{;a}\psi_{;b} - \frac{1}{2}(\eta(d\psi, d\psi) + m^2\psi^2)\eta_{ab}$. There is an analogue on any space-time $(M,g)$: $T_{ab} = \psi_{;a}\psi_{;b} - \frac{1}{2}(\psi_{;c}\psi_{;d}g^{cd} + m^2\psi^2)g_{ab}$.

**Exercise 2.20.** For the source-free Maxwell field $\widehat{\mathcal{L}}_m = -\frac{1}{16\pi} F_{ab} F_{cd} g^{ac} g^{bd}$, show that the method above yields

$$\begin{aligned}
&\int_M g(h,T)\,dv_g \\
&= \int_M -\frac{1}{8\pi}\Big( - F_{ab}F_{cd}g^{ai}h_{ij}g^{jd} - F_{ab}F_{cd}g^{ac}g^{bi}h_{ij}g^{jd} + \frac{1}{2}h_{ij}g^{ij}F_{ab}F_{cd}g^{ac}g^{bd}\Big)\,dv_g \\
&= \int_M \frac{1}{8\pi}h_{ij}\Big(F_{ab}F_{cd}g^{ai}g^{jc}g^{bd} + F_{ab}F_{cd}g^{ac}g^{bi}g^{jd} - \frac{1}{2}F_{ab}F_{cd}g^{ac}g^{bd}g^{ij}\Big)\,dv_g \\
&= \int_M h_{ij}\frac{1}{4\pi}\Big(F_{ab}F_{cd}g^{bd}g^{ai}g^{jc} - \frac{1}{4}F_{ab}F_{cd}g^{ac}g^{bd}g^{ij}\Big)\,dv_g
\end{aligned}$$

where in the last step we used the antisymmetry of $F_{ab}$. Thus

$$T_{ab} = \frac{1}{4\pi}\Big(F_{ac}F_{bd}g^{cd} - \frac{1}{4}F_{ij}F_{k\ell}g^{ik}g^{j\ell}g_{ab}\Big).$$

We now derive the Einstein equation from the *Ansatz* that the action for the combination of gravitation with the fields is simply the sum of a constant multiple of $\mathcal{R}_\Lambda$ with the action for the matter. In particular, we are assuming that the fields do not themselves appear in the Lagrangian for the gravitational field. With $\kappa = \frac{8\pi G}{c^4}$ as above, we consider the action

$$\mathcal{S}(g,\Psi) = \int_M \Big[\frac{1}{2\kappa}(R - 2\Lambda) + \widehat{\mathcal{L}}_m(g,\Psi)\Big]\,dv_g.$$

An important consideration in the earlier derivation of the Einstein tensor is that the stress-energy tensor is divergence-free. This can be derived from the variational definition above, along with *diffeomorphism invariance* of the action. The Einstein-Hilbert action $\mathcal{R}_\Lambda$ is clearly diffeomorphism invariant: if $\phi$ is a diffeomorphism of $M$, then $\mathcal{R}_\Lambda(g) = \mathcal{R}_\Lambda(\phi^* g)$. The assumption that the action for the matter $\mathcal{R}_m(g,\Psi) = \int_M \widehat{\mathcal{L}}_m(g,\Psi)\,dv_g$ is diffeomorphism invariant is, then, by the preceding observation equivalent to the diffeomorphism invariance of $\mathcal{S}(g,\Psi)$.

Now, suppose that $X$ is a smooth vector field on $M$, which generates a local one-parameter subgroup of diffeomorphisms $\phi_t$. Recall that $\frac{d}{dt}\Big|_{t=0}\phi_t^* g = L_X g$, where $(L_X g)_{ij} = X_{i;j} + X_{j;i}$.

**Exercise 2.21.** Suppose $X$ is compactly supported away from the boundary, and let $h = L_X g$. Show directly from the formula for the variation of the Einstein-Hilbert action and the Bianchi identities that $\frac{d}{dt}\Big|_{t=0}\mathcal{R}(g+th) = 0$.

For $X$ compactly supported away from the boundary with one-parameter group of diffeomorphisms $\phi_t$, we then have by (2.9) and (2.10) and the symmetry of $T$,

$$\begin{aligned} 0 &= \frac{d}{dt}\Big|_{t=0} \int_M \widehat{\mathcal{L}}_m(\phi_t^* g, \phi_t^* \Psi)\, dv_g \\ &= \frac{1}{2}\int_M g(L_X g, T)\, dv_g = \int_M X_{i;j} T^{ij}\, dv_g = -\int_M X_i T^{ij}_{\ ;j}\, dv_g. \end{aligned}$$

Since this holds for any compactly supported $X$, we see that $T$ must be divergence-free as desired.

2.3.4.1. *Related variational problems.* In this section, let $(M, g)$ be a closed manifold of dimension $n \geq 3$. We want to characterize critical points of $\mathcal{R}$ constrained to metrics of fixed volume, say $\mathrm{Vol}(g) = 1$. We let $\mathcal{M}$ be the set of (smooth) Riemannian metrics, and let $\mathcal{M}_1 \subset \mathcal{M}$ be the subset of unit volume metrics. Note that if $g \in \mathcal{M}$, and if $C > 0$ is a constant, then $dv_{(Cg)} = C^{n/2} dv_g$, and so $\mathrm{Vol}(Cg) = C^{n/2}\mathrm{Vol}(g)$. Thus $(\mathrm{Vol}(g))^{-2/n} g \in \mathcal{M}_1$. We also note that $R(Cg) = C^{-1}R(g)$.

We can consider $\mathcal{R}$ on $\mathcal{M}_1$, or by rescaling $g \in \mathcal{M}$ to $\bar{g} = (\mathrm{Vol}(g))^{-2/n} g \in \mathcal{M}_1$, we can consider the functional

$$\overline{\mathcal{R}}(g) = \mathcal{R}(\bar{g}) = \frac{\mathcal{R}(g)}{(\mathrm{Vol}(g))^{1-\frac{2}{n}}}.$$

**Proposition 2.22.** *A metric $g \in \mathcal{M}$ is critical for $\overline{\mathcal{R}}$ if and only if $g$ is Einstein. Thus $g \in \mathcal{M}_1$ is critical for $\mathcal{R}$ restricted to $\mathcal{M}_1$ if and only if $g$ is Einstein.*

**Proof.** Let $g_t = g + th$, and let $\bar{g}_t = (\mathrm{Vol}(g_t))^{-2/n} g_t \in \mathcal{M}_1$. Then $\overline{\mathcal{R}}(g_t) = \mathcal{R}(\bar{g}_t) = \frac{\mathcal{R}(g_t)}{(\mathrm{Vol}(g_t))^{1-\frac{2}{n}}}$. We compute, using $\frac{d}{dt}\Big|_{t=0}\mathrm{Vol}(g_t) = \int_M \frac{1}{2}\mathrm{tr}_g(h)\, dv_g$,

$$\begin{aligned} &\frac{d}{dt}\Big|_{t=0}\mathcal{R}(\bar{g}_t) \\ &= -\frac{\int_M h \cdot (\mathrm{Ric}(g) - \frac{1}{2}R(g)g)\, dv_g}{(\mathrm{Vol}(g))^{1-\frac{2}{n}}} - (1 - \frac{2}{n})\frac{\mathcal{R}(g)}{(\mathrm{Vol}(g))^{2-\frac{2}{n}}} \cdot \int_M \frac{1}{2}\mathrm{tr}_g(h)\, dv_g \\ &= -\frac{1}{(\mathrm{Vol}(g))^{1-\frac{2}{n}}}\int_M h \cdot \Big(\mathrm{Ric}(g) - \frac{1}{2}R(g)g + (\frac{n-2}{2n})(\mathrm{Vol}(g))^{-1}\mathcal{R}(g)g\Big)\, dv_g. \end{aligned}$$

For this to hold for all symmetric $(0,2)$-tensors $h$, we must have $\mathrm{Ric}(g) - \frac{1}{2}R(g)g + (\frac{n-2}{2n})(\mathrm{Vol}(g))^{-1}\mathcal{R}(g)g = 0$. From here, we can apply Lemma 2.12 to conclude $g$ is Einstein, but we can also proceed as follows. Taking the trace yields

$$\frac{n-2}{2}(R(g) - (\mathrm{Vol}(g))^{-1}\mathcal{R}(g)) = 0.$$

Hence $R(g) = (\mathrm{Vol}(g))^{-1}\mathcal{R}(g)$, a constant. From here, we clearly have $\mathrm{Ric}(g) = \frac{R(g)}{n}g$, and $R(g)$ is constant. □

We note that while the unconstrained problem had critical points characterized by $\mathrm{Ric}(g) = 0$, the constrained problem has a more general critical point equation, which can be interpreted as a Lagrange multiplier condition. If we interpret the gradient (in an $L^2$ integral sense) of $\mathcal{R}$ as $-(\mathrm{Ric}(g) - \frac{1}{2}R(g)g)$, then gradient of the volume functional would be $\frac{1}{2}g$, and a Lagrange multiplier condition would be of the form $-(\mathrm{Ric}(g) - \frac{1}{2}R(g)g) = \lambda \cdot \frac{1}{2}g$. From here, Lemma 2.12 would imply the Einstein condition.

**Remark 2.23.** Einstein metrics arise as stationary points of *normalized Ricci flow*. From (2.8), we see that a solution $g = g_t$ to the normalized Ricci flow $\frac{\partial g}{\partial t} = -2\mathrm{Ric}(g) + \frac{2R(g)}{n}g = -2(\mathrm{Ric}(g) - \frac{R(g)}{n}g)$ has volume $\mathrm{Vol}(g_t)$ constant.

We could further constrain the variation so that the metrics under consideration are not only of unit volume, but are also *pointwise conformal* to $g$, that is the metrics are of the form $f \cdot g$, where $f > 0$ is a smooth function on $M$. One can break up $\mathcal{M}$ into equivalence classes $[g]$ of pointwise conformal metrics. As an immediate corollary of the proof of Proposition 2.22 we have the following.

**Proposition 2.24.** *A metric $g$ is critical for $\overline{\mathcal{R}}$ amongst variations $g_t \in [g]$ if and only if $R(g)$ is constant. A metric $g \in \mathcal{M}_1$ is critical for $\mathcal{R}$ amongst variations $g_t \in \mathcal{M}_1 \cap [g]$ if and only if $R(g)$ is constant.*

**Proof.** Let $g_t = f_t g$, $t \in I$, be smooth in $I \times M$, with $f_0 = 1$. Then $f_t$ is smooth, and $h = \frac{d}{dt}g_t = \frac{df_t}{dt}g$. Let $\varphi = \frac{df_t}{dt}\Big|_{t=0}$, which can be any smooth function on $M$, since we could let $f_t = 1 + t\psi$. Applying the argument in the proof of Proposition 2.22, with $h = \psi g$, we have that

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0}\overline{\mathcal{R}}(g_t) \\
&= -\frac{1}{(\mathrm{Vol}(g))^{1-\frac{2}{n}}}\int_M \psi g \cdot \Big(\mathrm{Ric}(g) - \frac{1}{2}R(g)g + (\frac{n-2}{2n})(\mathrm{Vol}(g))^{-1}\mathcal{R}(g)g\Big)\, dv_g \\
&= \frac{1}{(\mathrm{Vol}(g))^{1-\frac{2}{n}}} \cdot \frac{n-2}{n}\int_M \psi\Big(R(g) - (\mathrm{Vol}(g))^{-1}\mathcal{R}(g)\Big)\, dv_g.
\end{aligned}$$

Since this must hold for all $\psi$, we have have $R(g) = \mathrm{Vol}(g))^{-1}\mathcal{R}(g)$, and conversely. □

**2.3.5. The Gauss-Bonnet Theorem for Closed Surfaces.** Before we return to general relativity, we discuss how the analysis of the Einstein-Hilbert action yields the basic form of the Gauss-Bonnet Theorem.

**Theorem 2.25.** *Suppose that* $(\Sigma, g)$ *is an orientable, closed Riemannian surface with Euler characteristic* $\chi(M)$*. If* $K = K(g)$ *is the Gauss (sectional) curvature, then*

$$\int_{\Sigma} K \, dv_g = 2\pi\chi(\Sigma).$$

**Proof.** Any such surface $\Sigma$ is a sphere, or a torus, or a torus with some number $\gamma$ of handles attached (genus $\gamma$). A sphere has Euler characteristic 2, as can be seen using a tetrahedral triangulation. It is also not too hard to triangulate a torus and compute its Euler characteristic to be 0. Higher genus surfaces are obtained by doing a connected sum to a torus. We can view this as removing a triangle from a surface and a torus, and gluing along the edges of the triangle. Let's say the original surface had Euler characteristic $\chi$, and we know the torus has Euler characteristic 0. So we know the total alternating sum of vertices, edges and faces at the start for both surfaces is $\chi$. In the process of adding a handle, we lost two faces, three edges and three vertices. Thus adding a handle brings the Euler characteristic down by 2. Thus $\chi(\Sigma_\gamma) = 2 - 2\gamma$, $\gamma = 0, 1, 2, \ldots$.

As we proved earlier, the Einstein-Hilbert action $\mathcal{R}(g)$ is critical at *every* metric $g$ on a two-dimensional manifold. This means that $\mathcal{R}$ is *constant* on the space $\mathcal{M}$ of metrics: any two metrics $g_1, g_2 \in \mathcal{M}$ can be connected by a linear path $g_t = (1-t)g_1 + tg_2$, $0 \leq t \leq 1$, and we've seen that $\mathcal{R}(g_t)$ is constant in $t$. Since $R(g) = 2K(g)$, we see that $\int_{\Sigma} K(g) \, dv_g$ is independent of $g$. Thus to compute it, we just need to pick a particularly nice $g$. For a sphere, we take a round metric for a unit sphere, so that $K = 1$ and $\mathrm{Vol}(g) = 4\pi$. The Euler characteristic of the sphere is 2, which can be seen by using the triangulation induced by a homeomorphism with a tetrahedron. For a torus, we have Euler characteristic 0, and we can use a flat metric on the torus to compute $\mathcal{R}(g)$. Consider a surface $S = S_1 \cup S_2$ which is homeomorphic to a sphere, and is obtained by smoothly capping off the ends of a circular cylinder. The cylinder is flat ($K = 0$). Divide it along a circular geodesic into two surfaces $S_1$ and $S_2$ with the geodesic circle as their common boundary. We can certainly do this in a symmetric manner, but to make a point, suppose we had not imposed symmetry. We could take either surface and apply a Euclidean motion to produce a new surface $(S_i', g')$, so that $S_i \cup S_i'$ is homeomorphic to a sphere, with $S_i \cap S_i'$

the same common geodesic circle, and with overall metric $\tilde{g}$. Thus

$$\int_{S_i} K(g)\, dv_g = \int_{S_i'} K(g')\, dv_{g'} = \frac{1}{2} \int_{S_i \cup S_i'} K(\tilde{g})\, dv_{\tilde{g}} = 2\pi. \tag{2.11}$$

Given a surface $\Sigma$, we consider $\Sigma'$ to be the connected sum of $\Sigma$ and a torus $\mathbb{T}$. We can readily put a metric $g$ on $\Sigma$ which has a region isometric to a surface $S_1$ as above with the cylindrical part extended a bit further past $S_1$. A similar statement is true for a metric $g_0$ on the torus, say with $S_2$ for which the cylindrical piece has the same radius as that of the cylindrical piece of $S_1$. Then $\Sigma' = (\Sigma \setminus S_1) \cup (\mathbb{T} \setminus S_2)$, and we can take a metric $g'$ on $\Sigma'$ compatible with the metric on each piece. If the Gauss-Bonnet Theorem holds for $\Sigma$, then by (2.11)

$$\int_{\Sigma'} K)(g')\, dv_g' = \int_{\Sigma} K(g)\, dv_g + \int_{\mathbb{T}} K(g_0)\, dv_{g_0} - 4\pi = \chi(\Sigma) - 2 \cdot 2\pi.$$

Since adding a handle contributed to a decrease in the Euler characteristic by 2, the Gauss-Bonnet Theorem follows by induction. □

## 2.4. Space-time examples

In this section we consider some examples of space-times and the form of the Einstein equation which they satisfy.

**2.4.1. Constant curvature space-times.** Let $\mathbb{R}^n_\nu$ for $0 \le \nu \le n$ denote the semi-Riemannian manifold $\mathbb{R}^n$ with the metric $\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \rangle = \epsilon_i \delta_{ij}$, where $\epsilon_i = -1$ for $1 \le i \le \nu$, and $\epsilon_i = 1$ for $\nu + 1 \le i \le n$. Recall the following manifolds defined as level sets of certain quadratic polynomials, with the metric induced from the indicated inclusions: for $n \ge 2$ and $r > 0$, we define

$$\mathbb{S}^n_1(r) = \{x \in \mathbb{R}^{n+1} : -(x^0)^2 + (x^1)^2 + \cdots + (x^{n+1})^2 = r^2\} \subset \mathbb{R}^{n+1}_1 = \mathbb{M}^{1+n}$$
$$\mathbb{H}^n_1(r) = \{x \in \mathbb{R}^{n+1} : -(x^0)^2 - (x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = -r^2\} \subset \mathbb{R}^{n+1}_2.$$

It is easy to see that $\mathbb{S}^n_1(r)$ is diffeomorphic to $\mathbb{R} \times \mathbb{S}^{n-1}$, which is simply connected for $n \ge 3$, while $\mathbb{H}^n_1(r)$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^{n-1}$. We have the following proposition.

**Proposition 2.26.** *A complete, simply connected $n$-dimensional Lorentzian manifold of constant curvature $C$ is isometric to one of the following:*

- $\mathbb{S}^n_1(r) \quad (n \ge 3,\ C = \frac{1}{r^2})$
- $\widetilde{\mathbb{S}}^2_1(r) \quad (n = 2,\ C = \frac{1}{r^2})$
- $\mathbb{R}^n_1 \quad (C = 0)$
- $\widetilde{\mathbb{H}}^n_1(r) \quad (C = -\frac{1}{r^2})$

$\mathbb{S}^4_1(r)$ is *de Sitter* space-time of curvature $C = \frac{1}{r^2}$, $\mathbb{H}^4_1(r)$ is *anti-de Sitter* space-time of curvature $C = -\frac{1}{r^2}$, while the universal cover $\widetilde{\mathbb{H}}^4_1(r)$ is *universal anti-de Sitter* space-time.

Constant curvature manifolds are Einstein. Indeed, in a constant curvature Lorentz four-manifold $(\mathcal{S}^4, g)$, we have $\mathrm{Ric}(g) = \frac{R(g)}{4}\, g$, where $R(g)$ is constant. Then we have $(\mathrm{Ric}(g) - \frac{1}{2}R(g)g) + \frac{1}{4}R(g)g = 0$. One can interpret this as Einstein's equation with $T = 0$ and $\Lambda = \frac{1}{4}R(g)$. Thus we have $\mathrm{Ric}(g) = \Lambda g$, and in the examples above, $\Lambda > 0$ corresponds to de Sitter space-time, while $\Lambda < 0$ corresponds to anti-de Sitter space-time.

**2.4.2. The Einstein Static Universe.** Consider the space $\mathbb{R} \times \mathbb{S}^3$ with the product metric $g = -c^2dt^2 + g_{\mathbb{S}^3}$, where $g_{\mathbb{S}^3}$ is the unit round metric on the three-sphere.

**Exercise 2.27.** Show that the Ricci curvature of this metric is $\mathrm{Ric}(g) = 2g_{\mathbb{S}^3}$.

From the exercise we see $R(g) = 6$, so that the Einstein tensor is $\mathrm{Ric}(g) - \frac{1}{2}R(g)g = 3c^2dt^2 - g_{\mathbb{S}^3}$, and $G_\Lambda(g) = (3-\Lambda)c^2dt^2 + (\Lambda-1)g_{\mathbb{S}^3}$. We identify this with the stress-energy tensor of a perfect fluid, with fluid velocity $\mathbb{U} = \frac{\partial}{\partial t}$, density $\rho$ and pressure $p$. If we write $T$ as a $(0,2)$-tensor, we have $T = (\rho + p)c^2dt^2 + pg = \rho c^2 dt^2 + pg_{\mathbb{S}^3}$. The Einstein equation $G_\Lambda(g) = \frac{8\pi G}{c^4}T$ is then equivalent to

$$3 - \Lambda = \frac{8\pi G}{c^4}\rho$$
$$\Lambda - 1 = \frac{8\pi G}{c^4}p.$$

Note that for $p \geq 0$, we must have $\Lambda \geq 1$. Note that $(\rho + p) = \frac{2}{\kappa}$ for $\kappa = \frac{8\pi G}{c^4}$.

**2.4.3. Friedmann-Lemaitre-Robinson-Walker.** Let $I \subset \mathbb{R}$ and let $\Sigma$ be a three-manifold. We consider a warped product metric of the form $g = -c^2dt^2 + (cf(t))^2 g_0$, where $g_0$ is a Riemannian metric of constant curvature $k$ on $\Sigma$, and $f$ is a positive function on $I$. Such metrics arise in cosmological models which incorporate the observation that the universe seems to be everywhere isotropic (with respect to a class of "observers," which may be on the galactic scale), which in turn implies spatial homogeneity of the manifolds orthogonal to the trajectories of such observers (spaces of simultaneity). See the references Carroll, O'Neill and Wald for more discussion of how isotropy and spatial homogeneity are translated into a metric of the above form.

Let $\mathbb{U} = \frac{\partial}{\partial t}$, and let $X$, $Y$ and $Z$ denote tangent vectors to $\Sigma_t = \{t\} \times \Sigma$ for $t \in I$. Recall the curvature convention $R(X, Y, Z) = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z$. We also denote $g(X, Y) = \langle X, Y\rangle$.

**Exercise 2.28.** Verify the following curvature formulas:

$$
\begin{aligned}
R(X,Y,Z) &= c^{-2}\Big[\Big(\frac{f'(t)}{f(t)}\Big)^2 + \frac{k}{(f(t))^2}\Big]\,[\langle Y,Z\rangle X - \langle X,Z\rangle Y] \\
R(X,\mathbb{U},\mathbb{U}) &= -\frac{f''(t)}{f(t)}X \\
R(X,Y,\mathbb{U}) &= 0 \\
R(X,\mathbb{U},Y) &= -c^{-2}\frac{f''(t)}{f(t)}\langle X,Y\rangle \mathbb{U}.
\end{aligned}
$$

We can now readily find the Ricci and scalar curvatures of $g$. It is easy to see

$$
\begin{aligned}
\mathrm{Ric}(\mathbb{U},\mathbb{U}) &= -3\frac{f''(t)}{f(t)} = 3\frac{f''(t)}{f(t)}\langle \mathbb{U},\mathbb{U}\rangle c^{-2} \\
\mathrm{Ric}(\mathbb{U},X) &= 0 \\
\mathrm{Ric}(X,Y) &= \Big[2\Big(\frac{f'(t)}{f(t)}\Big)^2 + 2\frac{k}{(f(t))^2} + \frac{f''(t)}{f(t)}\Big]\langle X,Y\rangle c^{-2} \\
R(g) &= 6\Big[\Big(\frac{f'(t)}{f(t)}\Big)^2 + \frac{k}{(f(t))^2} + \frac{f''(t)}{f(t)}\Big]c^{-2}.
\end{aligned}
$$

The stress-energy tensor that corresponds to this metric can be found using the Einstein equation: $T = \frac{c^4}{8\pi G}\big[\mathrm{Ric}(g) - \frac{1}{2}R(g)g + \Lambda g\big]$.

**Exercise 2.29.** Verify the following formulas.

$$
\begin{aligned}
T(\mathbb{U},X) &= 0 \\
T(X,Y) &= -\frac{c^2}{8\pi G}\left[\Big(\frac{f'(t)}{f(t)}\Big)^2 + \frac{k}{(f(t))^2} + 2\frac{f''(t)}{f(t)} - \Lambda c^2\right]\langle X,Y\rangle \\
T(\mathbb{U},\mathbb{U}) &= \frac{c^4}{8\pi G}\left[3\Big(\frac{f'(t)}{f(t)}\Big)^2 + 3\frac{k}{(f(t))^2} - \Lambda c^2\right].
\end{aligned}
$$

We can try to identify this with a perfect fluid, which has stress tensor $T = c^{-2}(\rho + p)\mathbb{U}^\flat \otimes \mathbb{U}^\flat + pg$. For instance, $T(\mathbb{U},\mathbb{U}) = c^2\rho$, $T(\mathbb{U},X) = 0$ and $T(X,Y) = p\langle X,Y\rangle$. We can identify the stress-energy tensor of the warped product as that of a perfect fluid, with

$$\begin{aligned}
\rho &= \frac{c^2}{8\pi G}\left[3\Big(\frac{f'(t)}{f(t)}\Big)^2 + 3\frac{k}{(f(t))^2} - \Lambda c^2\right] \\
&= \frac{c^4}{8\pi G}\left[3\Big(\frac{f'(t)}{cf(t)}\Big)^2 + 3\frac{k}{(cf(t))^2} - \Lambda\right] \\
p &= -\frac{c^2}{8\pi G}\left[\Big(\frac{f'(t)}{f(t)}\Big)^2 + \frac{k}{(f(t))^2} + 2\frac{f''(t)}{f(t)} - \Lambda c^2\right] \\
&= \frac{c^4}{8\pi G}\left[-\Big(\frac{f'(t)}{cf(t)}\Big)^2 - \frac{k}{(cf(t))^2} - 2\frac{f''(t)}{c^2 f(t)} + \Lambda\right].
\end{aligned}$$

Note that $\frac{4\pi G}{c^4}(\rho + 3p) - \Lambda = -3\frac{f''(t)}{f(t)}$. It is also elementary to derive the following equation from the above; it is basically the vanishing of the $t$-component of $\mathrm{div}_g T$, and so expresses conservation of energy: $\rho'(t) = -3(\rho(t) + p(t))\frac{f'(t)}{f(t)}$.

Observe that in the case when $f(t)$ is constant, for example in the Einstein static universe, then the geometry of the spatial slices is not dynamic. In this case, if $\Lambda = 0$, then $\rho$ and $p$ must have opposite signs, which is not appealing in terms of standard physics.

The Friedmann cosmological models are the cases of the above that correspond to dust models, so that $p = 0$, with $f'/f$ positive for some time $t_0 \in I$. This would model situations when the energy density dominates pressure, as might be the present situation in the universe (but not, say, near the Big Bang). When $p = 0$, we obtain the first-order linear equation $\rho'(t) + 3\rho(t)\frac{f'(t)}{f(t)} = 0$, so by integrating we obtain $\rho f^3$ is constant, say $\mathfrak{m}$. Substituting into the equation for $\rho$ above, we get the *Friedmann equation*

$$\frac{8\pi G\mathfrak{m}}{3c^2} \cdot \frac{1}{f(t)} = (f'(t))^2 + k - \frac{1}{3}\Lambda c^2 (f(t))^2.$$

Note that there is a critical value of $\Lambda$ for which one can achieve a static model (constant $f(t)$) when $p = 0$, given by $\Lambda c^2 = k^3\left(\frac{4\pi G\mathfrak{m}}{c^2}\right)^{-2}$.

We let $\Lambda = 0$ and $A = \frac{8\pi G\mathfrak{m}}{3c^2}$, so that the equation becomes

$$\frac{A}{f(t)} = (f'(t))^2 + k.$$

One can solve this for each possible sign of $k$. For $k = 0$, we get, assuming $f(t) > 0$ and $f'(t) \neq 0$, $\sqrt{f(t)}f'(t) = A_0$, so $(f(t))^{3/2} = A_0 t + A_1$. If we let $f(t) = 0$ at $t = 0$, we obtain $f(t) = Ct^{2/3}$. This describes a universe that expands from an initial Big Bang singularity (note that $f'(t) \to +\infty$ as $t \to 0^+$). If $k = 1$, the solution graph can be written in parametrized form as $t = \frac{1}{2}A(u - \sin u)$, $f = \frac{1}{2}A(1 - \cos u)$, which describes a cycloid. The geometry here expands from $f(0) = 0$ to a maximum value, then recollapses at $t = \pi A$ ($u = 2\pi$), the "Big Crunch." Similarly, when $k = -1$, $(f'(t))^2 = 1 + \frac{A}{f(t))} > 1$, so that $f(t)$ keeps growing without bound, and the spatial slices expand in size over time. We note that you can parametrize the solution graph as $t = \frac{1}{2}A(\sinh u - u)$, $f = \frac{1}{2}A(\cosh u - 1)$.

**2.4.4. Schwarzschild space-time.** Consider the vacuum Einstein equation $\mathrm{Ric}(g) = 0$. This is a nonlinear system of second-order partial differential equations for the metric components. One can reduce this to a second-order ordinary differential equation by imposing an *Ansatz* of spherical symmetry. One can argue (Birkhoff's Theorem, see Hawking-Ellis or Wald) that there is a time-like Killing field orthogonal to the orbits of the isometry group, and that the metric has the form (where $g_{\mathbb{S}^2}$ is the unit round metric on the two-sphere)

$$g = -f(r)dt^2 + h(r)dr^2 + r^2 g_{\mathbb{S}^2}.$$

By imposing the Einstein equations, one may compute to find $\frac{d}{dr}(fh) = 0$, by rescaling the $t$ variable we can arrange $fh = 1$. We then would find $\frac{d}{dr}(rf) = 1$. We can integrate to solve for $f$ and $h$, and hence the metric $g$. Then integrating the second equation we get $f(r) = 1 - \frac{2Gm}{c^2 r} = (h(r))^{-1}$. The notation for the constant of integration may seem odd, but it turns out there is good reason to identify the constant $m$ as the *mass* of the space-time. Indeed, the metric can represent the gravitational field in the exterior of a non-rotating spherically symmetric massive body, and in the weak field regime (large $r$), the effect of the gravitational field on test particles (ascertained by finding the geodesics in Schwarzschild) is roughly that of a Newtonian gravitational field for a point mass $m$.

We let

$$\bar{g}_S = -(1 - \frac{2Gm}{c^2 r})\ c^2\ dt^2 + (1 - \frac{2Gm}{c^2 r})^{-1}\ dr^2 + r^2 g_{\mathbb{S}^2}.$$

For simplicity, we take units for which $G = 1$ and $c = 1$, so that

$$\bar{g}_S = -(1 - \frac{2m}{r})dt^2 + (1 - \frac{2m}{r})^{-1}dr^2 + r^2 g_{\mathbb{S}^2}.$$

It appears that the metric is singular at $r = 2m$ and at $r = 0$. It turns out the geometry can be extended past $r = 2m$, but that as $r \to 0^+$, the square norm of the Riemann tensor blows up.

2.4.4.1. *Kruskal-Szekeres coordinates.* We fix a point on the sphere, and study the light cones as $r \to 2m^+$. Null vectors of the form $a\frac{\partial}{\partial t} + b\frac{\partial}{\partial r}$ must satisfy $\frac{a}{b} = \pm(1 - \frac{2m}{r})^{-1}$, which approaches infinity as $r \to 2m$. Thus the light cones seem to be pinching in these coordinates. To see how the metric is not ill-behaved at $r = 2m$, we use Kruskal-Szekeres coordinates, in which the light cones are better behaved. Define new coordinates for the $t$-$r$ space given by, for $r > 2m$,

$$u = (\frac{r}{2m} - 1)^{1/2} e^{\frac{r}{4m}} \cosh(\frac{t}{4m})$$
$$v = (\frac{r}{2m} - 1)^{1/2} e^{\frac{r}{4m}} \sinh(\frac{t}{4m}),$$

while for $r < 2m$,

$$u = (1 - \frac{r}{2m})^{1/2} e^{\frac{r}{4m}} \sinh(\frac{t}{4m})$$
$$v = (1 - \frac{r}{2m})^{1/2} e^{\frac{r}{4m}} \cosh(\frac{t}{4m}).$$

In either case, $u^2 - v^2 = e^{\frac{r}{2m}}(\frac{r}{2m} - 1)$, and the coordinates are singular at $r = 2m$, which corresponds to $u = v$. We can convert the metric for $r > 2m$ using the formulas

$$du = \frac{1}{4m} e^{\frac{r}{4m}} (\frac{r}{2m} - 1)^{1/2} \sinh(\frac{t}{4m})\, dt + \frac{1}{4m} e^{\frac{r}{4m}} (\frac{r}{2m} - 1)^{-1/2} \frac{r}{2m} \cosh(\frac{t}{4m})\, dr$$
$$dv = \frac{1}{4m} e^{\frac{r}{4m}} (\frac{r}{2m} - 1)^{1/2} \cosh(\frac{t}{4m})\, dt + \frac{1}{4m} \sqrt{u^2 - v^2} (\frac{r}{2m} - 1)^{-1} \frac{r}{2m} \sinh(\frac{t}{4m})\, dr.$$

Thus

$$\begin{aligned} dv^2 - du^2 &= \frac{1}{(4m)^2} e^{\frac{r}{2m}} (\frac{r}{2m} - 1)\, dt^2 - \frac{1}{(4m)^2} e^{\frac{r}{2m}} \frac{r}{2m} (1 - \frac{2m}{r})^{-1}\, dr^2 \\ &= \frac{1}{(4m)^2} e^{\frac{r}{2m}} \frac{r}{2m} \left[ (1 - \frac{2m}{r})\, dt^2 - (1 - \frac{2m}{r})^{-1}\, dr^2 \right] \end{aligned}$$

where we used $(1 - \frac{2m}{r})^{-1} = \frac{\frac{r}{2m}}{\frac{r}{2m} - 1}$. Thus we see

$$\bar{g}_S = -\frac{32m^3}{r} e^{-\frac{r}{2m}} (dv^2 - du^2) + r^2 g_{\mathbb{S}^2}.$$

This metric is not singular at $r = 2m$, which we see is a null hypersurface (its normal vector is null). Moreover, the coordinate representation of the light cones in these coordinates is uniform.

2.4.4.2. *Isotropic coordinates.* The spatial slice at constant $t$ in the above Schwarzschild metric are actually conformally flat. To see this, we just need to perform a change in the radial variable, from $r$ to $\tilde{r}$, with the change of coordinate an increasing function. We have

$$(1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{\mathbb{S}^2} = \left( (1 - \frac{2m}{r})^{-1/2} \frac{dr}{d\tilde{r}} \right)^2 d\tilde{r}^2 + \left( \frac{r}{\tilde{r}} \right)^2 \tilde{r}^2 g_{\mathbb{S}^2}.$$

We thus want to arrange

$$(1 - \frac{2m}{r})^{-1/2} \frac{dr}{d\tilde{r}} = \frac{r}{\tilde{r}}.$$

This becomes $\int \frac{d\tilde{r}}{\tilde{r}} = \int \frac{dr}{\sqrt{r^2-2mr}} = \int \frac{dr}{\sqrt{(r-m)^2-m^2}}$. We can integrate this with the substitution $r - m = m \cosh w$: $w = \log \tilde{r} + \widetilde{C}$, so $e^w = C\tilde{r}$. Thus $r = m + m\cosh w = m + \frac{m}{2}(C\tilde{r} + \frac{1}{C\tilde{r}})$. Thus $\frac{r}{\tilde{r}} = \frac{mc}{2} + \frac{m}{\tilde{r}} + \frac{m}{2c\tilde{r}^2}$. If we let $C = 2/m$, then we get

$$\frac{r}{\tilde{r}} = \left(1 + \frac{m}{2\tilde{r}}\right)^2.$$

Thus the Schwarzschild metric can also be written

$$\bar{g}_S = -\frac{(1-\frac{m}{2\tilde{r}})^2}{(1+\frac{m}{2\tilde{r}})^2}\, dt^2 + \left(1 + \frac{m}{2\tilde{r}}\right)^4 (d\tilde{r}^2 + \tilde{r}^2 g_{\mathbb{S}^2}).$$

Of course the Euclidean metric $g_E$ on $\mathbb{R}^3$ is written in standard spherical coordinates as $g_E = d\tilde{r}^2 + \tilde{r}^2\ g_{\mathbb{S}^2}$, so we see that the constant time slice is conformally Euclidean.

**2.4.5. The Kerr Metric.** The Kerr family of metrics is a family of Ricci-flat metrics that includes the Schwarzschild metrics, but allows for metrics which do not have full spherical symmetry, but rather axisymmetry. Such metrics model the exterior of a rotating gravitational object. We will write down the metric in a certain coordinate system, the Boyer-Lindquist coordinates. We let $r$, $\phi$ and $\theta$ be spherical coordinates on three-space. We will use the mathematical convention that $\theta$ is the polar angle, whereas $\phi$ is the angle between the position vector and the $z$-axis—-in most physics books, the angle notation is reversed, so be careful when comparing. In any case, the metric appears somewhat complicated, and depends on two parameters, $m$ and $a$. For simplicity, we use units where $G = 1$ and $c = 1$; in general, we can replace "$m$" with "$Gm/c^2$" and "$dt$" with "$c\,dt$" in what follows to obtain the more general formulas.)

We let $\Delta = r^2 - 2mr + a^2$ and $\rho^2 = r^2 + a^2\cos^2\phi$. Then the Kerr metric is given by

$$g = -\left(1 - \frac{2mr}{\rho^2}\right) dt^2 - a\frac{2mr\sin^2\phi}{\rho^2}(dt \otimes d\theta + d\theta \otimes dt) \tag{2.12}$$

$$+ \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\phi}{\rho^2}\sin^2\phi\, d\theta^2 + \frac{\rho^2}{\Delta}\, dr^2 + \rho^2\, d\phi^2. \tag{2.13}$$

This metric is *stationary*: $\frac{\partial}{\partial t}$ is a time-like Killing vector, but unlike with the Schwarzschild solution, if $a \neq 0$, this Killing field is not orthogonal to the constant $t$ slices. The axisymmetry is apparent, as $\frac{\partial}{\partial \theta}$ is also a Killing vector. One can define suitably the angular momentum and show that it is $J = am$ in the $z$-direction. Note that when $a = 0$, the metric reduces to the Schwarzschild metric.

General relativity predicts that a rotating massive object will cause the space-time around it to be "warped." One such manifestation of this is the *frame dragging effect*, which we illustrate in the Kerr metric.

**Lemma 2.30.** *If $X$ is a Killing field on $(M, g)$, and if $\gamma$ is a geodesic, then $\langle X, \gamma' \rangle$ is constant.*

**Proof.** $\gamma(\tau)$ has constant speed, so $\langle \nabla_X \gamma', \gamma' \rangle = 0$. By the geodesic equation then, $\frac{d}{d\tau}\langle X, \gamma' \rangle = \langle \nabla_{\gamma'} X, \gamma' \rangle = g_{ij}(\gamma^k)' X^i{}_{;k}(\gamma^j)' = X_{j;k}(\gamma^k)'(\gamma^j)' = X_{k;j}(\gamma^k)'(\gamma^j)'$. We see this must vanish by applying the Killing equation $X_{j;k} + X_{k;j} = 0$. □

First, consider in any space-time a time-like geodesic parametrized by proper time $\tau$, with velocity vector $\mathbb{U}(\tau)$. In the Kerr metric, $X = \frac{\partial}{\partial\theta}$ is a Killing vector, so that by the preceding lemma, $\mathbb{U}_\theta = \langle \mathbb{U}, \frac{\partial}{\partial\theta} \rangle$ is conserved along the geodesic. One can see this directly from the geodesic equations $\mathbb{U}^\alpha \mathbb{U}_{\alpha;\beta} = 0$, which can be written

$$\begin{aligned}\frac{d\mathbb{U}_\beta}{d\tau} &= \Gamma^\gamma_{\alpha\beta}\mathbb{U}^\alpha\mathbb{U}_\gamma = \frac{1}{2}g^{\gamma\nu}(g_{\nu\beta,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu})\mathbb{U}^\alpha\mathbb{U}_\gamma \\ &= \frac{1}{2}(g_{\nu\beta,\alpha} + g_{\alpha\nu,\beta} - g_{\alpha\beta,\nu})\mathbb{U}^\alpha\mathbb{U}^\nu = \frac{1}{2}g_{\alpha\nu,\beta}\mathbb{U}^\alpha\mathbb{U}^\nu.\end{aligned}$$

Since the metric is independent of $\theta$, then $\mathbb{U}_\theta$ is conserved. If the geodesic represents the path of a particle of rest mass $m_0$, then $\mathbb{P}_\theta = m_0\mathbb{U}_\theta$ is a component of the momentum one-form which is conserved.

Now consider a geodesic in Kerr with $\mathbb{U}_\theta = \langle \mathbb{U}, \frac{\partial}{\partial\theta} \rangle = 0$. Although $\mathbb{U}$ remains orthogonal to $\frac{\partial}{\partial\theta}$, the fact that $g_{t\theta} \neq 0$ means that $\mathbb{U}^\theta$ will be non-zero. In fact, we have $\mathbb{U}^t = \frac{dt}{d\tau} \neq 0$ and $\mathbb{U}^\theta = \frac{d\theta}{d\tau}$. Thus

$$\frac{d\theta}{dt} = \frac{\mathbb{U}^\theta}{\mathbb{U}^t} = \frac{g^{\theta\alpha}\mathbb{U}_\alpha}{g^{t\alpha}\mathbb{U}_\alpha} = \frac{g^{\theta t}}{g^{tt}} = -\frac{g_{\theta t}}{g_{\theta\theta}} = \frac{2mra}{(r^2 + a^2)^2 - a^2\Delta\sin^2\phi}.$$

Although in the metric the trajectory is orthogonal to the direction of the symmetry given by the Killing field $\frac{\partial}{\partial\theta}$, the coordinate description of the motion the trajectory has a component in the $\theta$-direction. Thus a freely falling object seems to pick up some coordinate angular momentum in the direction of rotation of the massive object, say, generating this gravitational field (metric).

*Chapter 3*

# The Einstein Constraint Equations

## 3.1. Introduction

Many physical models admit an initial value formulation. In Newtonian mechanics, for instance, if we suppose the force is a function of the positions of the various particles under observation, then Newton's Second Law gives a system of ordinary differential equations which in principle will yield the evolution of the system once the initial positions and velocities of the particles are specified. The wave and heat equations also admit initial value problems. Maxwell's equations likewise admit an initial value formulation, but unlike the previous examples, where the initial configuration is essentially unconstrained, one cannot arbitrarily prescribe the electric and magnetic field at $t = 0$, say, and hope to solve Maxwell's equations. The reason is that parts of Maxwell's equations do not involve time derivatives: the spatial divergence of $\mathbf{B}$ vanishes, as does that of $\mathbf{E}$ (in the source-free case). These two divergence constraints place restrictions on the vector fields one can use to prescribe initial data. As it turns out, these are the only constraints.

In this chapter, we discuss an initial value formulation for Einstein's equation. Geometrically, the initial data will be a three-manifold $\Sigma$, endowed with a Riemannian metric $g$ and symmetric $(0, 2)$-tensor $K$. That this three-manifold embeds into a Lorentzian manifold $(M, \bar{g})$ satisfying the Einstein equations, with induced metric $g$ and second fundamental form $K$ imposes constraints on $g$ and $K$. These are the *Einstein constraint equations*, the study of solutions to which form an interesting and rich subject for geometric analysis.

**3.1.1. Maxwell's Equations.** We briefly discuss the initial value problem for the source-free Maxwell equations on Minkowski space-time $\mathbb{M}^4 = (\mathbb{R}^4, \eta)$. For simplicity we take $c = 1$, so $x^0 = t$. We saw that the Maxwell equations can be written in terms of the Faraday tensor $F$, an antisymmetric $(2, 0)$-tensor with corresponding two-form $F^\flat$. In inertial coordinates we write $F^\flat = (E_1 dx^1 + E_2 dx^2 + E_3 dx^3) \wedge dx^0 + (B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2)$. Maxwell's equations can then be written $\mathrm{div}_\eta F = 0$, $dF^\flat = 0$. From the second equation and the Poincaré Lemma, we can write $F^\flat = dA$ for a one-form $A = A_\mu dx^\mu$, well-defined up to a gauge function $\phi$: $F^\flat = d(A + d\phi)$ for any smooth $\phi$. If $\vec{\nabla}$ is the spatial (Euclidean) gradient operator, and if we let $\mathbf{A} = A_1 \frac{\partial}{\partial x^1} + A_2 \frac{\partial}{\partial x^2} + A_3 \frac{\partial}{\partial x^3}$, then we can identify $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$, while $\mathbf{E} = \vec{\nabla} A_0 - \frac{\partial \mathbf{A}}{\partial t}$. Note that the spatial divergence vanishes automatically: $\vec{\nabla} \cdot \mathbf{B} = 0$. The spatial divergence of $\mathbf{E}$ is given by $\vec{\nabla} \cdot \mathbf{E} = \Delta A_0 - \vec{\nabla} \cdot \frac{\partial \mathbf{A}}{\partial t} = 0$, where $\Delta$ is the Euclidean Laplacian on $\mathbb{R}^3$. As it turns out, this equation is equivalent to $(\mathrm{div}_\eta F)^0 = F^{0\nu}{}_{;\nu} = 0$. Now, Maxwell's equations should be second order in $A$, but this one component only has first derivatives of $A$ in it. This component of Maxwell's equations, equivalent to the vanishing of the spatial divergence of $\mathbf{E}$, must be satisfied by the initial data. This imposes a constraint on the data.

To formulate a second-order initial value problem for $A$, we can use the gauge freedom in $A$. The *Lorentz gauge condition* imposes precisely that $0 = \mathrm{div}_\eta A = -\frac{\partial A_0}{\partial t} + \vec{\nabla} \cdot \mathbf{A}$. Clearly, Maxwell's equations are equivalent to $\Box A = -\frac{\partial^2 A_\mu}{\partial t^2} + \Delta A_\mu = 0$, *along with* the Lorentz gauge condition.

**Exercise 3.1.** Verify this last claim. While you're at it, show that $\vec{\nabla} \cdot \mathbf{E} = 0$ is equivalent to $(\mathrm{div}_\eta F)^0 = F^{0\nu}{}_{;\nu} = 0$. Use the fact that $F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}$.

Let's now proceed. We specify initial data $A_\mu$ and $\frac{\partial A_\mu}{\partial t}$ at $t = 0$. We can actually arrange the Lorentz gauge condition at $t = 0$ by adding $d\phi$ to $A$, for a function $\phi$ depending only on $(x^1, x^2, x^3)$; this involves solving a Poisson equation on $\mathbb{R}^3$, and we assume we are working in function spaces where we can solve this equation (for instance, where the fields decay sufficiently near infinity), and similar comments apply in the remainder of this section. This effects $A_\mu$ at $t = 0$ by a gauge transformation, but it doesn't change the time derivative at $t = 0$. We now solve the wave equation $\Box A_\mu = 0$. We will have produced a solution to Maxwell's equations, *provided* we can show that the gauge condition is propagated in time. How do we do this? We haven't yet incorporated the constraint. In fact, the condition $\vec{\nabla} \cdot \mathbf{E} = 0$, i.e. $\Delta A_0 - \vec{\nabla} \cdot \frac{\partial \mathbf{A}}{\partial t} = 0$ at $t = 0$, together with the wave equation for $A_0$, yields $\frac{\partial}{\partial t}\Big|_{t=0} (\mathrm{div}_\eta A) = 0$. Now, the wave equation for each $A_\mu$ also implies $\Box(\mathrm{div}_\eta A) = 0$. The gauge condition at $t = 0$ together with the constraint

$\vec{\nabla} \cdot \mathbf{E} = 0$ implies that $\mathrm{div}_\eta A$ satisfies the wave equation with zero initial data. Thus $\mathrm{div}_\eta A = 0$ for all $t$, and the gauge condition propagates in time.

We can also formulate the initial value problem in terms of $\mathbf{E}$ and $\mathbf{B}$, with the constraints that the spatial divergences vanish. We can proceed by re-writing the problem in the above form. Namely, we solve for $A$ as above, where the initial data for $A$ is as follows: $A_0 = 0$, $\vec{\nabla} \times \mathbf{A} = \mathbf{B}$ (possible to find $\mathbf{A}$ since $\vec{\nabla} \cdot \mathbf{B} = 0$ at $t = 0$; at $t = 0$, set $\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}$, and for the Lorentz gauge condition, we take $\frac{\partial A_0}{\partial t} = \vec{\nabla} \cdot \mathbf{A}$ at $t = 0$. We can now solve as above to produce solutions to the Maxwell equations with given initial electric and magnetic fields.

We can use a different gauge condition as well. The source-free Maxwell's equations imply vector wave equations for $\mathbf{E}$ and $\mathbf{B}$. We can thus solve for $\mathbf{E}$ in time using the initial values for $\mathbf{E}$ as well as imposing the Maxwell equation $\frac{\partial \mathbf{E}}{\partial t} = \vec{\nabla} \times \mathbf{B}$ at $t = 0$. The vector wave equation for $\mathbf{E}$ implies $\vec{\nabla} \cdot \mathbf{E}$ solves the wave equation. Moreover, $\frac{\partial}{\partial t}(\vec{\nabla} \cdot \mathbf{E}) = \vec{\nabla} \cdot (\vec{\nabla} \times \mathbf{B}) = 0$ at $t = 0$, and thus the constraint $\vec{\nabla} \cdot \mathbf{E} = 0$ propagates in time. To handle the other constraint, we solve for $\mathbf{A}$ at $t = 0$ so that $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$. We take $A_0 = 0$. The evolution of $\mathbf{A}$ is then given by $\frac{\partial \mathbf{A}}{\partial t} = -\mathbf{E}$. From this and the definition $\mathbf{B} = \vec{\nabla} \times \mathbf{A}$, we obtain the Maxwell equation $\frac{\partial \mathbf{B}}{\partial t} = -\vec{\nabla} \times \mathbf{E}$. We now have

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{E}}{\partial t} - \vec{\nabla} \times \mathbf{B}\right) &= \frac{\partial^2 \mathbf{E}}{\partial t^2} - \vec{\nabla} \times \left(\vec{\nabla} \times \frac{\partial \mathbf{A}}{\partial t}\right) \\
&= \frac{\partial^2 \mathbf{E}}{\partial t^2} + \vec{\nabla} \times (\vec{\nabla} \times \mathbf{E}) \\
&= \frac{\partial^2 \mathbf{E}}{\partial t^2} + \vec{\nabla}(\vec{\nabla} \cdot \mathbf{E}) - \Delta \mathbf{E} = 0
\end{aligned}$$

by the wave equation for $\mathbf{E}$ and the constraint $\vec{\nabla} \cdot \mathbf{E} = 0$. From the propagation of the constraint, we see that $\vec{\nabla} \cdot \mathbf{A}$ is constant in time. We can replace $\mathbf{A}$ by $\mathbf{A} + \vec{\nabla}\phi$ for a suitable time-independent $\phi$ to arrange $\vec{\nabla} \cdot \mathbf{A} = 0$ at $t = 0$, and hence for all time. This is the *Coulomb gauge*.

**3.1.2. The Gauss and Codazzi Equations.** Assume $\Sigma^k$ is a submanifold of $(M^n, \bar{g})$ on which $\bar{g}$ induces a metric $g$ (Riemannian or Lorentzian) on $\Sigma$. We let $\bar{g}(X, Y) = \langle X, Y \rangle$.

**Example 3.2.** Consider the forward light cone minus the origin $\Sigma^+ = \{x^0, x^1, x^2, x^3) : x^0 = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}\} \subset \mathbb{M}^4$. Consider a point $p$ in this submanifold at which $y = 0 = z$. Then the tangent space $T_p\Sigma^+$ is spanned by $\left\{\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}\right\}$. Note that $\frac{\partial}{\partial x^0} - \frac{\partial}{\partial x^1}$ is orthogonal to all of $T_p\Sigma^+$, and so the Minkowski metric does not induce a metric on $\Sigma^+$.

We can decompose the tangent bundle of $M$ into the direct sum of the tangent and normal bundles of $\Sigma$: $TM = T\Sigma \oplus N\Sigma$. It is not hard to show that the Levi-Civita connection $\nabla^\Sigma$ from the induced metric on $\Sigma$ satisfies, for smooth vector fields $X$ and $Y$ tangent to $\Sigma$,

$$\nabla_X Y = \nabla^\Sigma_X Y + I\!I(X,Y)$$

where $\nabla^\Sigma_X Y$ is tangent to $\Sigma$, and $I\!I(X,Y)$ is normal to $\Sigma$.

**Lemma 3.3.** *Suppose $\Sigma \subset M$ is a Lorentzian or Riemannian submanifold. For any vector fields $X$, $Y$ tangent to $\Sigma$, the induced Levi-Civita connection $\nabla^\Sigma_X Y$ is the tangential projection of $\nabla_X Y$, while $I\!I(X,Y)$ is tensorial in $X$ and $Y$, and is symmetric.*

**Proof.** We define $\nabla^\Sigma_X Y$ as the tangential component of $\nabla_X Y$, and we show it satisfies the defining properties of the Levi-Civita connection. Clearly $\nabla^\Sigma$ is torsion-free, since $\nabla_X Y - \nabla_Y X = [X,Y]$, which is tangential to $\Sigma$; this latter equation also implies that $I\!I(X,Y) = I\!I(Y,X)$. Clearly $\nabla^\Sigma_X Y$ is $C^\infty$-linear in $X$, $\mathbb{R}$-linear in $Y$, since $\nabla_X Y$ has these properties. Since $\nabla_X(fY) = X(f)Y + f\nabla_X Y$, and $X(f)Y$ is tangential to $\Sigma$, by taking projections we get the corresponding equation for $\nabla^\Sigma$. Finally, the preceding equation also implies that $I\!I(X,Y)$ is $C^\infty$-linear in $Y$, and hence by symmetry in $X$ as well. □

**Definition 3.4.** The tensor $I\!I$ is the (vector-valued) *second fundamental form.* The *mean curvature vector field* is $\mathbf{H} = \mathrm{tr}_g I\!I = \sum_{i=1}^{k} \epsilon_i I\!I(E_i,E_i)$, where $\{E_1,\ldots,E_k\}$ is an orthonormal basis of $T_p\Sigma$ with $\epsilon_i = \langle E_i, E_i\rangle$. The scalar-valued second fundamental form with respect to the unit normal vector $n$ is given by $K(X,Y) = \langle I\!I(X,Y), n\rangle$, and the respective mean curvature $H$ is the trace: $H = \mathrm{tr}_g K$.

In the case of a hypersurface, we let $\epsilon = \langle n, n\rangle = \pm 1$, and $I\!I(X,Y) = \epsilon K(X,Y)n$, while $\mathbf{H} = \epsilon H n$.

We begin by reviewing the proof of the Gauss equation, which relates the curvature of the submanifold to the ambient curvature and the second fundamental form.

**Proposition 3.5.** *Suppose $\Sigma$ is a submanifold of $(M,g)$. For any $X, Y, Z, W \in T_p\Sigma$, then we have*

$$\begin{aligned}\langle R^\Sigma(X,Y,Z), W\rangle &= \langle R(X,Y,Z), W\rangle - \langle I\!I(X,Z), I\!I(Y,W)\rangle \\ &\quad + \langle I\!I(X,W), I\!I(Y,Z)\rangle. \end{aligned} \tag{3.1}$$

**Proof.** We decompose the following into tangential and normal components, and discard inner products between any tangential and normal vectors. We also note that if $X$ and $Y$ are tangential and $N$ is normal, then $\nabla_X\langle Y,N\rangle = 0$, which is equivalent to $\langle \nabla_X Y, N\rangle = -\langle Y, \nabla_X N\rangle$.

$$\begin{aligned}
\langle R(X,Y,Z),W\rangle &= \langle \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z, W\rangle\\
&= \langle \nabla_X(\nabla^\Sigma_Y Z + I\!I(Y,Z)) - \nabla_Y(\nabla^\Sigma_X Z + I\!I(X,Z)) - \nabla^\Sigma_{[X,Y]}Z, W\rangle\\
&= \langle \nabla^\Sigma_X\nabla^\Sigma_Y Z + I\!I(X,\nabla^\Sigma_Y Z) + \nabla_X(I\!I(Y,Z))\\
&\quad - \nabla^\Sigma_Y\nabla^\Sigma_X Z - I\!I(Y,\nabla^\Sigma_X Z) - \nabla_Y(I\!I(X,Z)) - \nabla^\Sigma_{[X,Y]}Z, W\rangle\\
&= \langle R^\Sigma(X,Y,Z),W\rangle - \langle I\!I(Y,Z),\nabla_X W\rangle + \langle I\!I(X,Z),\nabla_Y W\rangle\\
&= \langle R^\Sigma(X,Y,Z),W\rangle - \langle I\!I(Y,Z),I\!I(X,W)\rangle + \langle I\!I(X,Z),I\!I(Y,W)\rangle.
\end{aligned}$$

The last equation followed by decomposing $\nabla_X W$ and $\nabla_Y W$ into tangential and normal components. □

To derive the Einstein constraint equations, we will use the Einstein equatio n, together with the Gauss equation, and the Codazzi equation, which we present now. We first define the normal connection $\nabla^\perp$ in the normal bundle $N\Sigma$ as follows: for $V$ tangent to $\Sigma$ and $Z$ a normal vector field to $\Sigma$, we define $\nabla^\perp_V Z$ to be the normal component of $\nabla_V Z$. We can use this connection (and impose a product rule) to differentiate tensors with values in the normal bundle, in particular the second fundamental form: for $V$, $X$ and $Y$ tangent to $\Sigma$,

$$(\nabla_V I\!I)(X,Y) := \nabla^\perp_V(I\!I(X,Y)) - I\!I(\nabla^\Sigma_V X, Y) - I\!I(X,\nabla^\Sigma_V Y).$$

For $X$, $Y$ and $Z$ tangent to $\Sigma$, let $R^\perp(X,Y,Z)$ be the normal component of $R(X,Y,Z)$.

**Proposition 3.6.** *For $X$, $Y$ and $Z$ tangent to $\Sigma$,*

$$R^\perp(X,Y,Z) = (\nabla_X I\!I)(Y,Z) - (\nabla_Y I\!I)(X,Z). \tag{3.2}$$

**Proof.** As in the proof of the Gauss equation, we decompose the curvature tensor:

$$\begin{aligned}
R(X,Y,Z) &= \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\\
&= \nabla_X(\nabla^\Sigma_Y Z + I\!I(Y,Z)) - \nabla_Y(\nabla^\Sigma_X Z + I\!I(X,Z)) - \nabla_{[X,Y]}Z.
\end{aligned}$$

Thus by taking the normal component we have, using $[X,Y] = \nabla^\Sigma_X Y - \nabla^\Sigma_Y X$,

$$\begin{aligned} R^\perp(X,Y,Z) &= I\!I(X, \nabla^\Sigma_Y Z) + \nabla^\perp_X (I\!I(Y,Z)) \\ &\quad - I\!I(Y, \nabla^\Sigma_X Z) - \nabla^\perp_Y (I\!I(X,Z)) - I\!I([X,Y],Z) \\ &= I\!I(X, \nabla^\Sigma_Y Z) + \nabla^\perp_X (I\!I(Y,Z)) \\ &\quad - I\!I(Y, \nabla^\Sigma_X Z) - \nabla^\perp_Y (I\!I(X,Z)) - I\!I(\nabla^\Sigma_X Y, Z) + I\!I(\nabla^\Sigma_Y X, Z) \\ &= (\nabla_X I\!I)(Y,Z) - (\nabla_Y I\!I)(X,Z). \end{aligned}$$

□

## 3.2. The Einstein Constraint Equations

Suppose $(M, \bar{g})$ is Lorentzian, and $\Sigma \subset M$ is a three-dimensional space-like hypersurface, i.e. the induced metric $g$ on $\Sigma$ is Riemannian. Let $n$ be a (local) time-like unit vector field to $\Sigma$. We let $T$ be a symmetric $(0,2)$-tensor and assume that $(M, \bar{g})$ satisfies the Einstein equation $G_\Lambda(\bar{g}) = \kappa T$. We let $\mathbb{J} = (-T(n,\cdot))^\sharp$, so that $\mathbb{J}^\nu = -T^{\mu\nu} n_\mu$. We write $\mathbb{J} = \rho n + J^\sharp$, where $J^\sharp$ is tangent to $\Sigma$. Then $\rho = T(n,n) = T_{00}$, and if $E_1, E_2, E_3$, is a local frame for $T\Sigma$ (and we let $E_0 = n$ to complete the indexing), we write $J^\sharp = \sum_{i=1}^{3} J^i E_i$, with $-T_{\mu j} n^\mu = -T_{0j} = -T(n, E_j) = \sum_{i=1}^{3} J^i g_{ij} = J_j$, $j = 1,2,3$. Note that $J^i = \sum_{j=1}^{3} g^{ij} J_j = -T^{i\mu} n_\mu = T^{i0}$, $i = 1,2,3$. Then $\rho$ is the energy density of the matter fields as measured by the observer with four-velocity $cn$, and $J$ is ($c$ times) the corresponding momentum density one-form. We note if $n$ is future-pointing, then the Dominant Energy Condition ($\mathbb{J}$ is future-pointing causal) implies $\rho \geq |J|_g = \sqrt{\sum_{i=1}^{3} J^i J_i}$.

**Theorem 3.7.** *The following system of equations must hold on* $\Sigma$.

$$R(g) - \|K\|_g^2 + H^2 = \frac{16\pi G}{c^4}\rho + 2\Lambda = 2\kappa\rho + 2\Lambda \tag{3.3}$$

$$\mathrm{div}_g K - d(\mathrm{tr}_g K) = \frac{8\pi G}{c^4} J = \kappa J. \tag{3.4}$$

**Proof.** Let $E_1$, $E_2$, $E_3$ be a local orthonormal frame field for $\Sigma$. We first apply the Gauss equation:

$$\begin{aligned}\sum_{i,j=1}^{3}\langle R(E_i,E_j,E_j),E_i\rangle &= \sum_{i,j=1}^{3}\Big[\langle R^{\Sigma}(E_i,E_j,E_j),E_i\rangle - \langle I\!I(E_j,E_j), I\!I(E_i,E_i)\rangle \\ &\qquad + \langle I\!I(E_i,E_j), I\!I(E_i,E_j)\rangle\Big] \\ &= R(g) - \|K\|_g^2 + H^2.\end{aligned}$$

Because $\langle n,n\rangle = -1$, we have

$$\operatorname{Ric}(E_i,E_i) = -\langle R(n,E_i,E_i),n\rangle + \sum_{j=1}^{3}\langle R(E_i,E_j,E_j),E_i\rangle,$$

so that with the Einstein tensor $G(\bar g) = \operatorname{Ric}(\bar g) - \frac{1}{2}R(\bar g)\bar g$,

$$\begin{aligned}\sum_{i,j=1}^{3}\langle R(E_i,E_j,E_j),E_i\rangle &= \operatorname{Ric}(n,n) + \sum_{i=1}^{3}\operatorname{Ric}(E_i,E_i) \\ &= R(\bar g) + 2\operatorname{Ric}(n,n) = 2G(n,n) \\ &= (-2\Lambda\bar g + \kappa T)(n,n) = 2\Lambda + 2\kappa\rho\end{aligned}$$

For (3.4), we apply the Codazzi equation (3.2). For any $W$ tangent to $\Sigma$,

$$\sum_{i=1}^{3} R^{\perp}(W,E_i,E_i) = \sum_{i=1}^{3} -(\nabla_{E_i} I\!I)(W,E_i) + (\nabla_W I\!I)(E_i,E_i).$$

Since $\langle R(n,W,n),n\rangle = 0$ and $\langle R^{\perp}(W,E_i,E_i),n\rangle = \langle R(W,E_i,E_i),n\rangle$, the above becomes

$$\operatorname{Ric}(W,n) = \sum_{i=1}^{3}\langle -(\nabla_{E_i} I\!I)(W,E_i) + (\nabla_W I\!I)(E_i,E_i), n\rangle.$$

We evaluate the right-hand side first. Each term can be re-written in terms of $K$, starting with the first term:

$$\begin{aligned}-(\nabla_{E_i} I\!I)(W,E_i) &= -\Big[\nabla^{\perp}_{E_i}(I\!I(W,E_i)) - I\!I(\nabla^{\Sigma}_{E_i}W,E_i) - I\!I(W,\nabla^{\Sigma}_{E_i}E_i)\Big] \\ &= \nabla^{\perp}_{E_i}(K(W,E_i)n) - K(\nabla^{\Sigma}_{E_i}W,E_i)n - K(W,\nabla^{\Sigma}_{E_i}E_i)n \\ &= \nabla_{E_i}(K(W,E_i))n - K(\nabla^{\Sigma}_{E_i}W,E_i)n - K(W,\nabla^{\Sigma}_{E_i}E_i)n\end{aligned}$$

where we used the fact that $\langle \nabla_{E_i} n, n\rangle = 0$. Summing we obtain

$$\sum_{i=1}^{3} -(\nabla_{E_i} I\!I)(W,E_i) = ((\operatorname{div}_g K)(W))n.$$

For the second term, we may assume that we are computing in normal coordinates for $(\Sigma, g)$ at $p \in \Sigma$, and that $\{E_1, E_2, E_3\}$ is the coordinate frame. In particular, at $p$ we have $\nabla^\Sigma_W E_i = 0$ for all $i = 1, 2, 3$ and all $W \in T_p\Sigma$. Since the Christoffel symbols vanish at $p$, we also have $d(\mathrm{tr}_g K)\Big|_p = d\left[\sum_{i=1}^{3} K(E_i, E_i)\right]\Big|_p$. Using this, and again using $\langle \nabla_W n, n \rangle = 0$, we obtain at $p$

$$\begin{aligned}\sum_{i=1}^{3} (\nabla_W I\!I)(E_i, E_i) &= \sum_{i=1}^{3} \Big( \nabla^\perp (I\!I(E_i, E_i)) - 2I\!I(\nabla^\Sigma_W E_i, E_i) \Big) \\ &= \sum_{i=1}^{3} \nabla^\perp_W (I\!I(E_i, E_i)) \\ &= -(d(\mathrm{tr}_g K)(W))n.\end{aligned}$$

Thus we obtain $\mathrm{Ric}(W, n) = -\left((\mathrm{div}_g K) - d(\mathrm{tr}_g K)\right)(W)$. If we apply the Einstein equation, then since $\bar{g}(W, n) = 0$, we obtain $\mathrm{Ric}(W, n) = \kappa T(W, n) = -\kappa J(W)$, as desired. □

We make some remarks on these equations. Note that the constraints come from imposing $G_\Lambda(\bar{g})(n, \cdot) = \kappa T(n, \cdot)$, and as we saw above, this does not involve time derivatives of the metric. The *time-symmetric*, or Riemannian, case of the constraints is the case $K = 0$. In this case, (3.3) reduces to $R(g) = \kappa\rho + 2\Lambda$. In case $\Lambda = 0$, then, the scalar curvature $R(g)$ is proportional to the energy density, and $R(g) \geq 0$ if and only if $\rho \geq 0$. The *maximal* case is $H = 0$, so that $R(g) = \|K\|^2_g + \kappa\rho + 2\Lambda \geq \kappa\rho + 2\Lambda$. In the vacuum case ($T = 0$), then the time-symmetric constraints reduce to $R(g) = 2\Lambda$, and in the maximal case we have $R(g) \geq 2\Lambda$; we often consider $\Lambda = 0$, which highlights the conditions of zero or nonnegative scalar curvature of $(\Sigma, g)$.

## 3.3. The Initial Value Formulation for the Vacuum Einstein Equation

In this section we discuss aspects of the analysis and geometry of the initial value formulation for $\mathrm{Ric}(\bar{g}) = 0$, or more generally $\mathrm{Ric}(\bar{g}) = \Lambda\bar{g}$. We will follow the approach of the foundational work of Choquet-Bruhat. The purpose of this section is to illustrate the ideas; to be mathematically precise, we should specify function spaces and state carefully the relevant partial differential equations results that are in play here. We won't do this, but refer the reader to the text *General Relativity and the Einstein Equations* by Yvonne Choquet-Bruhat.

**3.3.1. Einstein's equation in the harmonic gauge.** In order to formulate the initial value problem as a nonlinear wave equation, we express the Einstein equations in terms of a partial differential equation *along with* a gauge condition, as we did for Maxwell's equations above. The gauge choice we use is the choice of coordinates. In fact, we will use *harmonic coordinates*, or in the Lorentzian case, *wave coordinates* $x^\alpha$, which are coordinates so that $\lambda^\alpha := \Box_{\bar{g}} x^\alpha = 0$. On any Lorentzian manifold we can locally set up wave coordinates: given any local coordinates $y^\mu$, we solve the Cauchy problem for the linear wave equations $\Box_{\bar{g}} x^\mu = 0$ with initial conditions $x^\mu = y^\mu$ and $\nabla_n x^\mu = \nabla_n y^\mu$ impose on a level surface of $y^0$ ($\frac{\partial}{\partial y^0}$ is time-like), where $n$ is the unit normal to the level surface.

Let $\Gamma^k_{ij} = \frac{1}{2}\bar{g}^{km}(\bar{g}_{im,j} + \bar{g}_{jm,i} - \bar{g}_{ij,m})$ be the Christoffel symbols for $\bar{g}$ in a coordinate system. It's easy to see

$$\lambda^\alpha = \Box_{\bar{g}} x^\alpha = \bar{g}^{ij} x^\alpha_{;ij} = \bar{g}^{ij}(-\Gamma^k_{ij} x^\alpha_{,k}) = -\bar{g}^{ij}\Gamma^\alpha_{ij}.$$

In what follows we will write "$A \sim B$" to mean $A - B$ is a function of the components $\bar{g}_{ij}$ and $\bar{g}_{ij,k}$; in particular, $A - B$ does not depend on second derivatives of the metric components. For example,

$$-(\bar{g}_{\alpha i}\lambda^\alpha_{,j} + \bar{g}_{\alpha j}\lambda^\alpha_{,i}) \sim \bar{g}_{\alpha i}\bar{g}^{km}\Gamma^\alpha_{km,j} + \bar{g}_{\alpha j}\bar{g}^{km}\Gamma^\alpha_{km,i}.$$

**Exercise 3.8.** Show that

$$\frac{1}{2}(\bar{g}_{\alpha i}\lambda^\alpha_{,j} + \bar{g}_{\alpha j}\lambda^\alpha_{,i}) \sim -\frac{1}{2}\bar{g}^{km}(\bar{g}_{ki,mj} + \bar{g}_{jm,ki} - \bar{g}_{km,ij}). \tag{3.5}$$

Now the components of the Ricci curvature of $\bar{g}$ are given by

$$R_{ij} = \Gamma^k_{ij,k} - \Gamma^k_{ik,j} + \Gamma^k_{k\ell}\Gamma^\ell_{ij} - \Gamma^k_{j\ell}\Gamma^\ell_{ik} \sim \Gamma^k_{ij,k} - \Gamma^k_{ik,j}.$$

Moreover,

$$\begin{aligned}\Gamma^k_{ij,k} - \Gamma^k_{ik,j} &\sim \frac{1}{2}\bar{g}^{km}[(\bar{g}_{im,jk} + \bar{g}_{jm,ik} - \bar{g}_{ij,mk}) - (\bar{g}_{im,kj} + \bar{g}_{km,ij} - \bar{g}_{ik,mj})] \\ &= \frac{1}{2}\bar{g}^{km}(\bar{g}_{jm,ik} - \bar{g}_{ij,mk} - \bar{g}_{km,ij} + \bar{g}_{ik,mj}).\end{aligned}$$

From this equation together with (3.5), we see

$$R^H_{ij} := R_{ij} + \frac{1}{2}(\bar{g}_{\alpha i}\lambda^\alpha_{,j} + \bar{g}_{\alpha j}\lambda^\alpha_{,i}) \sim -\frac{1}{2}\bar{g}^{mk}\bar{g}_{ij,mk}.$$

We now consider the reduced Einstein equation $R^H_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$, which can be written as a quasi-linear wave equation:

$$-\frac{1}{2}\bar{g}^{\alpha\beta}\bar{g}_{\mu\nu,\alpha\beta} + \Psi_{\mu\nu}((\bar{g}_{ij}), (\bar{g}_{ij,k})) = 0. \tag{3.6}$$

A solution to (3.6) will solve the Einstein equation if we can arrange $\lambda^\alpha = 0$ for $\alpha = 0, 1, 2, 3$. From the seminal work of Leray, along with a rescaling argument (see Wald, Ch. 10), we know we can solve a system like (3.6) for small time.

**3.3.2. The Einstein constraints and the propagation of the gauge condition.** Suppose we are given a solution $(\Sigma, g, K)$ of the Einstein constraint equations for $\mathrm{Ric}(\bar{g}) = \Lambda\bar{g}$. We prescribe initial data for (3.6). Choose local coordinates $x^i$, $i = 1, 2, 3$ on $U \subset \Sigma$. We will construct a solution on the product of an interval (the $x^0 = t$ interval) and $U$. For $\mu, \nu = 1, 2, 3$, let the initial date be given by $\bar{g}_{\mu\nu} = g_{\mu\nu}$, and $\bar{g}_{\mu\nu,0} = -2K_{\mu\nu}$. We also let $\bar{g}_{00} = -1$ and for $\mu = 1, 2, 3$, $\bar{g}_{0\mu} = 0$. For $\mu = 0, 1, 2, 3$, we will choose the initial values of $\bar{g}_{0\mu,0}$ to arrange the gauge condition, as we explain.

Now using the formula we derived earlier for the derivative of a determinant, we see

$$\lambda^\alpha = \frac{1}{\sqrt{|\det \bar{g}|}}(\sqrt{|\det \bar{g}|}\ \bar{g}^{\beta\gamma} x^\alpha_{,\gamma})_{,\beta} = \bar{g}^{\beta\alpha}_{,\beta} + \frac{1}{2}\bar{g}^{\beta\alpha}\bar{g}^{\rho\sigma}\bar{g}_{\rho\sigma,\beta}.$$

We get at $t = 0$,

$$\lambda^0 = \bar{g}^{00}_{,0} + \frac{1}{2}\bar{g}^{00}\bar{g}^{\rho\sigma}\bar{g}_{\rho\sigma,0} \tag{3.7}$$

and for $i = 1, 2, 3$, we have at $t = 0$,

$$\lambda^i = \bar{g}^{0i}_{,0} + \sum_{j=1}^{3}\left(\bar{g}^{ji}_{,j} + \frac{1}{2}\bar{g}^{ji}\bar{g}^{\rho\sigma}\bar{g}_{\rho\sigma,j}\right). \tag{3.8}$$

From (3.8), we see we can choose $\bar{g}_{0i,0}$ at $t = 0$ to arrange $\lambda^i\Big|_{t=0} = 0$ for $i = 1, 2, 3$. We can then substitute into (3.7) to determine $\bar{g}_{00,0}$ at $t = 0$ in order that $\lambda^0\Big|_{t=0} = 0$ as well.

The question now is how to guarantee that $\lambda^\alpha = 0$ propagates in time. As we will see, a homogeneous linear wave equation for $\lambda^\alpha$ is a consequence of the Bianchi identities, while the Einstein constraints will show that the initial time derivative of $\lambda^\alpha$ vanishes. Together with the preceding paragraph, this will allow us to conclude that the gauge conditions that we have arranged at $t = 0$ propagate in time.

We begin with the following simple exercise.

**Exercise 3.9.** Assuming $R^H_{\mu\nu} = \Lambda\bar{g}_{\mu\nu}$, show that

$$(G_\Lambda(\bar{g}))_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R(\bar{g})\bar{g}_{\mu\nu} + \Lambda\bar{g}_{\mu\nu} = -\frac{1}{2}\bar{g}_{\alpha\mu}\lambda^\alpha_{,\nu} - \frac{1}{2}\bar{g}_{\alpha\nu}\lambda^\alpha_{,\mu} + \frac{1}{2}\bar{g}_{\mu\nu}\lambda^\alpha_{,\alpha}.$$

The vacuum constraints that are satisfied are precisely $(G_\Lambda(\bar{g}))_{\mu\nu}n^\nu = 0$ for $\mu = 0, 1, 2, 3$, or in our set up, $(G_\Lambda(\bar{g}))_{\mu 0} = 0$ at $t = 0$. Note that by our arrangement of the gauge condition at $t = 0$, $\lambda^\mu_{,i} = 0$ for $i = 1, 2, 3$ and all $\mu$. The component of the vacuum constraint for $\mu = 0$ (at $t = 0$) is just (using the preceding remark)

$$0 = -\frac{1}{2}\bar{g}_{\alpha 0}\lambda^{\alpha}{}_{,0} - \frac{1}{2}\bar{g}_{\alpha 0}\lambda^{\alpha}{}_{,0} + \frac{1}{2}\bar{g}_{00}\lambda^{\alpha}{}_{,\alpha} = \frac{1}{2}\lambda^{0}{}_{,0}.$$

For $\mu = i = 1, 2, 3$, we have at $t = 0$ (again using the vanishing of the spatial derivatives of $\lambda^{\mu}$, and the initial condition on the metric)

$$0 = -\frac{1}{2}\bar{g}_{\alpha i}\lambda^{\alpha}{}_{,0} - \frac{1}{2}\bar{g}_{\alpha 0}\lambda^{\alpha}{}_{,i} + \frac{1}{2}\bar{g}_{i0}\lambda^{\alpha}{}_{,\alpha} = -\frac{1}{2}\sum_{j=1}^{3} g_{ji}\lambda^{j}{}_{,0}$$

Since this is true for $i = 1, 2, 3$ and the matrix $(g_{ij})$ is invertible, we have $\lambda^{j}{}_{,0}$ must vanish as well at $t = 0$.

**Exercise 3.10.** Use the preceding exercise to show that for a solution of (3.6), the vanishing of the divergence of $G_{\Lambda}(\bar{g})$ is equivalent to

$$0 = -\frac{1}{2}\bar{g}_{\alpha\nu}\Box_{\bar{g}}\lambda^{\alpha} + B^{\theta}_{\nu\gamma}\left(((\bar{g})_{\rho\sigma}), (\bar{g}_{\rho\sigma,\beta})\right)\lambda^{\gamma}{}_{,\theta}.$$

From this exercise we see that the partials of $\lambda^{\alpha}$ satisfy a homogeneous linear hyperbolic system with vanishing initial data. Thus $\lambda^{\alpha}$ vanish identically.

What we've discussed here is a local construction. We would like to say we can patch overlapping local solutions together, essentially proving a uniqueness result. Suppose $U \subset \Sigma$ corresponds to the common intersection of two such patches, and that $\mathcal{V}_1$ and $\mathcal{V}_2$ are the corresponding space-times generated by solving the initial-value problem as above. The Cauchy data both agree with that coming from $(U, g, K)$. It is not so hard (cf. Choquet-Bruhat's book, for example) to argue that there are diffemorphisms $f_i : \mathcal{U} \to \mathcal{U}_i \subset \mathcal{V}_i$ which yield wave coordinates, and in which the the Cauchy data $\bar{g}_{\mu\nu}$ and $\bar{g}_{\mu\nu,0}$ at $t = 0$ agree, where these are the components in these wave charts. Indeed one can find coordinates that yield the agreement of the Cauchy data components; one can modify (as we noted above) the coordinates to wave coordinates, by using the values of the coordinates and their time derivatives at $t = 0$ for the wave equation. We'll then have under pullback to $\mathcal{U}$ two solutions of the reduced Einstein equations in wave gauge, with the *same* initial data. Uniqueness for the Cauchy problem for the reduced Einstein equation in the harmonic gauge shows that the solutions must agree.

**3.3.3. Geometry of the evolution.** Consider a Lorentzian manifold $(I \times \Sigma, \bar{g})$, where $I$ is an interval, around 0, say, and where the *slices* $\Sigma_t = \{t\} \times \Sigma$ are space-like. We let $g = g(t)$ be the induced metric on $\Sigma_t$. Let $n$ be normal to the slices, parallel to the space-time gradient of $t$, pointing in the same time direction as $\frac{\partial}{\partial t} = Nn + X$, where $X$ is tangent to the slice, and $N > 0$. $N$ is the *lapse* function, and $X$ is the *shift* vector field. We can write the metric in local coordinates $x^i$, $i = 1, 2, 3$, for $\Sigma$ as $\bar{g}$ as

$$\bar{g} = -N^2\, dt^2 + g_{ij}(dx^i + X^i\, dt) \otimes (dx^j + X^j\, dt) \tag{3.9}$$

Also note that the Einstein summation convention here and in this section will be over the spatial indices from 1 to 3.

The first and second fundamental forms of the slices form a family of solutions to the Einstein constraint equations. In our discussion of solving the Einstein equations from initial data, we chose $N = 1$ and $X = 0$ on the initial slice. One can view the solution of the initial value problem as determining a lapse and shift for a space-time splitting. It is also possible (by work of Choquet-Bruhat) to formulate the evolution equations by suitably prescribing $N$ and $X$, and then solving for the induced metric and second fundamental forms of the slices. Given $N$ and $X$, we indicate here the evolution equations of these geometric quantities on the slices.

With our convention on $K$, we have $\nabla_X Y = \nabla^\Sigma_X Y - K(X,Y)n$, where $X$ and $Y$ are tangent to a slice. We suppress the "$t$" subscript on $\Sigma_t$.

We compute the time derivative of the induced metric. Let $e_i = \frac{\partial}{\partial x^i}$ be a coordinate frame for $\Sigma$, and let $e_0 = \frac{\partial}{\partial t}$. Using metric compatibility, the torsion-free property of the connection, and the fact that all the $e_\mu$ commute, we have

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial t} &= \bar{g}(\nabla_{e_i} e_0, e_j) + \bar{g}(e_i, \nabla_{e_j} e_0) \\
&= \bar{g}(\nabla_{e_i}(Nx + X), e_j) + \bar{g}(e_i, \nabla_{e_j}(Nn + X)) \\
&= N\bar{g}(\nabla_{e_i} n, e_j) + N\bar{g}(e_i, \nabla_{e_j} n) + \bar{g}(\nabla_{e_i} X, e_j) + \bar{g}(e_i, \nabla_{e_j} X) \\
&= -2NK_{ij} + g(\nabla^\Sigma_{e_i} X, e_j) + g(e_i, \nabla^\Sigma_{e_j} X) \\
&= -2NK_{ij} + (\mathcal{L}_X g)_{ij}
\end{aligned}$$

where $\mathcal{L}_X g$ is the Lie derivative, and it is not hard to show $(\mathcal{L}_X g)_{ij} = X_{i;j} + X_{j;i}$, where the semi-colon indicates covariant differentiation for the Levi-Civita connection of $g$. Note that we can solve for the second fundamental form: $K_{ij} = -\frac{1}{2} N^{-1} \left( \frac{\partial g_{ij}}{\partial t} - (\mathcal{L}_X g)_{ij} \right)$.

A more laborious exercise determines the time evolution of $K$.

**Exercise 3.11.** Show that

$$\frac{\partial K_{ij}}{\partial t} = -N_{;ij} + (\mathcal{L}_X K)_{ij} - N(R_{ij} - R^\Sigma_{ij} + 2K^\ell_i K_{j\ell} - K^\ell{}_\ell K_{ij})$$

where $R_{ij}$ are components of $\mathrm{Ric}(\bar{g})$, and $R^\Sigma_{ij}$ are components of $\mathrm{Ric}(g)$, and $(\mathcal{L}_X K)(Y,Z) = X(K(Y,Z)) - K([X,Y],Z) - K(Y,[X,Z])$ is the Lie derivative of $K$.