2013 Summer Graduate Workshop, Cortona, Italy:
Mathematical General Relativity
Some Basic Problems on Riemannian Geometry

Here are some basic problems from Riemannian geometry. The problems aren't supposed to be overly challenging, but rather a reader's guide for self-study, if you're trying to learn basics quickly, or trying to brush up. We use the Einstein summation convention throughout—sum over a pair of upper and lower repeated indices. Our convention for the Riemann curvature tensor agrees with that of John M. Lee's book, for instance (but is opposite in sign from that used in DoCarmo or O'Neill)—also watch the index convention:

$$R(X,Y,Z) = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$
$$R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k}\right) = R^{\ell}_{ijk} \frac{\partial}{\partial x^\ell}\,, \qquad R_{ijk\ell} = g_{m\ell} R^m_{ijk}.$$

From the definition of curvature, we immediately get the vector field version of the Ricci formula: $Z^i_{;jk} - Z^i_{;kj} = Z^\ell R^i_{kj\ell}$. The Ricci tensor in DoCarmo and Lee agree, which means the way they are defined from the Riemann tensor is slightly different to account for sign. In our convention,

$$\mathrm{Ric}(X,Y) = dx^i\left(R\left(\frac{\partial}{\partial x^i}, X, Y\right)\right) = g^{k\ell} g\left(R\left(\frac{\partial}{\partial x^k}, X, Y\right), \frac{\partial}{\partial x^\ell}\right)$$
$$R_{ij} = \mathrm{Ric}\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = R^\ell_{\ell ij}.$$

Problems.

1. If (M,g) is a Riemannian metric with Levi-Civita connection ∇. The *Hessian* of u is defined by $\mathrm{Hess}_g u = \nabla(du)$. It is a $(0,2)$-tensor. The *Laplacian* is the trace of the Hessian: $\Delta_g u = \mathrm{tr}_g(\mathrm{Hess}_g u)$.

Show that $\mathrm{Hess}_g u(X,Y) = Y[X[u]] - \nabla_Y X[u]$, where $X[u] = du(X)$ is the directional derivative of u in the direction X. Conclude that the Hessian is symmetric.

2. a. If T is a $(1,2)$-tensor field on (M,g). If the components of T in a coordinate system (x^i) are T^i_{jk}, i.e. $T = T^i_{jk} \frac{\partial}{\partial x^i} \otimes dx^j \otimes dx^k$, then the components $T^i_{jk;\ell}$ of ∇T satisfy

$$T^i_{jk;\ell} = T^i_{jk,\ell} + \Gamma^i_{m\ell} T^m_{jk} - \Gamma^m_{j\ell} T^i_{mk} - \Gamma^m_{k\ell} T^i_{jm}.$$

b. If ∇ is the Levi-Civita connection associated to (M,g), show that $g_{ij;k} = 0$ and ${g^{ij}}_{;k} = 0$.

3. Let $\gamma : I \to (M,g)$ be a smooth curve, and let $0 \in I$. Let $P_t : T_{\gamma(0)}M \to T_{\gamma(t)}M$ be the parallel transport operator. If V is a smooth vector field along γ, show that the covariant derivative $\frac{DV}{dt}\Big|_{t=0} = \frac{d}{dt}\Big|_{t=0} P_t^{-1}(V(t))$. (Hint: use an orthonormal parallel frame field $e_1(t), \dots, e_n(t)$ along γ.)

4. In this problem we will be considering connections ∇ on M (i.e. on the tangent bundle of M). The connections will not necessarily be the Levi-Civita connections coming from metrics.

a. If ∇ and $\widehat{\nabla}$ are two connections on M. Show that $S(X,Y) = \nabla_X Y - \widehat{\nabla}_X Y$ is tensorial in both X and Y (i.e. it is C^∞-linear in X and Y).

b. If ∇ is a connection on M, show that $\tau(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ is tensorial in X and Y. τ is the *torsion tensor*.

S and τ each determine a $(1,2)$-tensor, e.g. $(\theta, X, Y) \mapsto \theta(\tau(X,Y))$, where θ is a one-form.

c. For a connection ∇, we can define for any (smooth) function u on M, $\mathrm{Hess}(u) = \nabla(du) = \nabla^2 u$. Prove that ∇ is torsion-free if and only if $\mathrm{Hess}(u)$ is symmetric for all smooth u.

5. Let (M,g) be a Riemannian manifold with Levi-Civita connection ∇. For any vector field X, ∇X is a $(1,1)$ tensor: $(\theta, Y) \mapsto \theta(\nabla_Y X)$, whose components are $X^i_{\ ;j}$. The *divergence* of X is the contraction of ∇X; in coordinates: $\mathrm{div}_g(X) = X^i_{\ ;i}$. Recall also the *interior product*: if ω is a k-form and X is a vector, then $\iota_X(\omega) := \omega(X, \ldots)$ is a $(k-1)$-form. Furthermore, for each $p \in M$, there is a neighborhood U of p (e.g. a coordinate neighborhood), on which there is an n-form ω, *a local volume form*, which satisfies $\omega(e_1, \ldots, e_n) = \pm 1$ for any orthonormal basis $\{e_1, \ldots, e_n\}$ of $T_q M$, $q \in U$.

a. In a coordinate neighborhood $U \subset M$, one can perform Gram-Schmidt on the coordinate basis fields to produce a *smooth orthonormal frame field* $\{E_1, \ldots E_n\}$ on U. If $\theta^1, \ldots, \theta^n$ are the dual frame fields, show that $\omega = \theta^1 \wedge \cdots \wedge \theta^n$ is a local volume form. Show furthermore that in local coordinates, $\omega = \pm\sqrt{\det(g_{ij})}\, dx^1 \wedge \cdots \wedge dx^n$.

b. If ω is a (local) volume form on M, show that $d(\iota_X \omega) = \mathrm{div}_g(X)\, \omega$.

b. Assume M is oriented with global volume form ω. If ∂M is nonempty, give it the induced orientation with induced volume measure σ. If ν is the outward unit normal to the boundary of M, then show $\int\limits_M \mathrm{div}_g X \omega = \int\limits_{\partial M} g(X,\nu)\ \sigma$. Show that the analogous formula holds in the general case (even if M were not orientable) if the form ω is replaced by the volume measure $d\mu_g$ (and induced measure $d\sigma_g$ on the boundary), where in local coordinates $d\mu_g = \sqrt{\det(g_{ij})}\ dx$ where $dx = dx^1 \cdots dx^n$ is the usual measure on $\mathbb{R}^n$.

c. Let $\det g = \det(g_{ij})$. Show that $\Delta_g u = \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x^i}\left(g^{ij}\sqrt{\det g}\ \frac{\partial u}{\partial x^j}\right)$.

d. Suppose $\Delta_g u = -\lambda u$ for some nontrivial (smooth) function u on a closed Riemannian manifold (M,g). Show that $\lambda \geq 0$. In case $\lambda = 0$, what is u?

6. a. Prove the Ricci formula: if α is a one-form, then $\alpha_{i;jk} - \alpha_{i;kj} = \alpha_\ell R^\ell_{jki}$.

b. Use the Ricci formula to prove the following, for smooth functions u:

$$g(\mathrm{grad}_g(\Delta_g u), \nabla u) + |\mathrm{Hess}_g u|_g^2 + \mathrm{Ric}_g(\mathrm{grad}_g u, \mathrm{grad}_g u) = \frac{1}{2}\Delta_g(|\mathrm{grad}_g u|_g^2).$$

7. Let $E_1, \ldots, E_n$ be a local frame field with dual frame $\theta^1, \ldots, \theta^n$. Let ∇ be a connection on M. Since $\nabla_X Y$ is tensorial in X, there is a matrix of one-forms ω_i^j so that $\nabla_X E_i = \omega_i^j(X) E_j$. Furthermore, from the torsion tensor τ above, we construct torsion two-forms τ^j given by $\tau(X,Y) = \tau^j(X,Y) E_j$; clearly τ^j is alternating.

a. Prove Cartan's *first structural equation*: $d\theta^j = \theta^i \wedge \omega_i^{\ j} + \tau^j$.

REMARK: It might be useful to recall the following formula for the differential of a one-form α: $d\alpha(X,Y) = X[\alpha[Y]] - Y[\alpha[X]] - \alpha([X,Y])$.

b. Now suppose (M,g) is Riemannian, and ∇ is the Levi-Civita connection. In particular, $\tau = 0$. Let $\Omega_i^{\ j}$ be a matrix of two-forms defined by $\Omega_i^{\ j} = \frac{1}{2}R^j_{\ell k i}\theta^k \wedge \theta^\ell$. Prove Cartan's *second structural equation* $\Omega_i^{\ j} = d\omega_i^{\ j} - \omega_i^{\ k} \wedge \omega_k^{\ j}$.

8. Consider the surface of revolution in Euclidean space given by $\sqrt{x^2+y^2} = \cosh z$. The surface is an example of a *catenoid*. Compute its Gaussian curvature. Show that the mean curvature is zero (and hence the catenoid is *minimal*).

9. a. Prove that Euclidean space $(\mathbb{R}^3, g_E)$ does not admit a closed immersed minimal surface. To do this, show that there must be a point on the surface where the Gaussian curvature is strictly positive.

b. If you followed the hint, you immediately conclude that for any embedding of a two-torus $\mathbb{T}^2$ into Euclidean $\mathbb{R}^3$ is not flat. Show that there is an embedding of a flat torus into Euclidean $\mathbb{R}^4$.

10. Suppose $M^n \subset (R^{n+1}, g_E)$ is an oriented hypersurface in Euclidean space, oriented with a smooth unit normal vector field N, and with induced metric g (first fundamental form). Let $S(X) = -\nabla_X N$.

a. Show that for all $p \in M$, S gives a linear operator $S : T_pM \to T_pM$, *the shape operator.*

b. Prove $(\nabla_X S)(Y) = (\nabla_Y S)(X)$ for X and Y tangent to M, and thus $\mathrm{div}_g(S) = d(\mathrm{tr}_g(S))$ on M.

11. Let $\mathbb{X} : U \to \mathbb{R}^3$ be an embedding of an open subset U of $\mathbb{R}^2$ onto a surface $\Sigma = \mathbb{X}(U)$. The components of the induced metric on Σ are written $g_{11} = E = \mathbb{X}_u \cdot \mathbb{X}_u > 0$, $g_{12} = F = \mathbb{X}_u \cdot \mathbb{X}_v = g_{21}$, and $g_{22} = G = \mathbb{X}_v \cdot \mathbb{X}_v > 0$. Let $N = \frac{\mathbb{X}_u \times \mathbb{X}_v}{\|\mathbb{X}_u \times \mathbb{X}_v\|}$ be a smooth unit normal field. Let S be the corresponding shape operator (see the previous problem), represented by the matrix $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ in the basis $\{\mathbb{X}_u, \mathbb{X}_v\}$. Let the associated bilinear form $(X,Y) \mapsto S(X) \cdot Y = g(S(X), Y)$ be represented in the same basis by $\begin{pmatrix} \ell & m \\ m & n \end{pmatrix}$.

a. Relate these two matrices to the first fundamental form matrix.

b. Consider the equations $\mathbb{X}_{uuv} - \mathbb{X}_{uvu} = \mathbf{0}$, and $\mathbb{X}_{vuv} - \mathbb{X}_{vvu} = \mathbf{0}$. Decompose this into tangential and normal components. What do you get?

12. a. Let $M \subset (\bar{M}, \bar{g})$ be an embedding, with induced metric g on M. Show that the second fundamental form $II(X,Y) = (\nabla_X Y)^{\mathrm{nor}}$ vanishes identically on M if and only if for every v tangent to M, the $\bar{g}$-geodesic γ_v lies entirely in M. In this case we say M is *totally geodesic* in $\bar{M}$.

b. For Riemannian manifolds (M_1, g_1) and (M_2, g_2), we consider the product $(\bar{M} = M_1 \times M_2, \bar{g} = g_1 \oplus g_2)$. Show that $\{p_1\} \times M_2$ and $M_1 \times \{p_2\}$, for any $p_i \in M_i$, are totally geodesic in $\bar{M}$.