ADDITIONAL TOPICS AND PROBLEMS
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Let g_E be the Euclidean metric, given in Cartesian coordinates by $(g_E)_{ij} = \delta_{ij}$.

1. A special case of the Positive Mass Theorem.

A special case where a proof of the PMT is fairly simple is when the AF-manifold under consideration is conformally flat, and harmonically flat at infinity, i.e.:

- (M, g) is isometric to $(\mathbb{R}^3, u^4 g_E)$ where $u > 0$ is a smooth function $\mathbb{R}^3 \to \mathbb{R}$,
- u is harmonic outside a compact set.

Problem 1. Show that M as above has nonnegative scalar curvature if and only if $\Delta u \leq 0$ (Euclidean Laplacian). (Hint: Use the transformation law for the scalar curvature under conformal deformations.)

Problem 2. Suppose that $\lim_{|x| \to +\infty} u(x) = a$, so that (M, g) as above is asymptotically flat. Show that $u(x) = a + \frac{b}{|x|} + O(|x|^{-2})$. (Hint: Use spherical harmonics.)

Problem 3. Show that the ADM-mass of such M is $m = 2ab$.

Problem 4. Assume that M is as above, has nonnegative scalar curvature, and is such that $u \to 1$ as $|x| \to \infty$. (Why can we assume this?) Prove the Positive Mass Theorem on M, namely that that $m \geq 0$, with equality if and only if $u \equiv 1$. (Hint: Use the divergence theorem for a suitable flux integral involving u over a large coordinate sphere.)

Problem 5. Generalize the above proof to arbitrary dimensions.

(Open) Problem 6. What about the Penrose Inequality for these manifolds?

Partial work towards proving this has been carried out by Bray and Iga [*Comm. Anal. Geom. 10 (2002), no. 5, 999–1016.*], Schwartz [*Ann. Henri Poincaré 12 (2011), no. 1, 67–76.*], and Freire and Schwartz [*to appear in Comm. PDE.*].

2. Penrose Inequality for spherically symmetric metrics.

Hubert Bray included in his PhD thesis an interesting proof of the Riemannian Penrose Inequality for spherically symmetric, asymptotically flat 3-manifolds. It is explained in Sections 1.3 and 2.1 of his manuscript, available at arXiv:0902.3241 [math.DG]. The argument provides a good introduction to the subject.

Problem. Generalize Bray's proof to arbitrary dimensions.