MOTIVIC INTEGRATION

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Disclaimer: The reader is warned that the author is a believer in the white lie to make technical subjects more accessible. A more accurate survey of this topic is the paper of Looijenga: math.AG/0006220.

Motivic integration was introduced by M. Kontsevich in a 1995 lecture in Orsay, where he announced an affirmative solution to the following conjecture of Batyrev: *two birationally equivalent Calabi-Yau manifolds have the same Hodge numbers.* This conjecture was motivated by work in theoretical physics, namely string theory, which predicts that as a manifold the universe is locally a product of $\mathbb{R}^4$ (space-time) and a compact component (a Calabi-Yau manifold). The conjecture means that two such special manifolds share the same important numerical invariants, provided that they contain large enough isomorphic open subsets.

Kontsevich gave a remarkably elegant and conceptual proof of this result, essentially as a corollary of his theory of motivic integration. He was inspired by the theory of p-adic integration, which Batyrev himself had used to prove a weaker form of his conjecture.

Motivic Integration. There is an associated theory of motivic integration for each smooth complex variety. Like any integration theory, we need

(1) a space on which the integration theory is defined,
(2) a measure on (certain) subsets of that space,
(3) some interesting measurable functions, and
(4) an understanding of how integrals transform under change of coordinates.

The novelty of motivic integration is that the measures and functions take values not in the real numbers, but in a more exotic ring $\mathcal{M}$, called the *motivic ring.* The elements of $\mathcal{M}$, roughly speaking, are formal $\mathbb{Z}$-linear combinations of varieties with addition corresponding to disjoint union of varieties and multiplication corresponding to direct products of varieties. The ring $\mathcal{M}$ also contains formal multiplicative inverses of certain elements, much in the way the ring of rational numbers is obtained by adjoining formal multiplicative inverses of integers. This ring is not ordered, so properties of a measure such as "subadditive on unions" do not make sense. However, other basic features of measure theory, such as additivity on finite disjoint unions, do carry over. We discuss each of these four items below.

(1) The arc space of X. Fix a smooth complex variety X. Motivic integration theory for X is an integration theory on the *arc space* $J_\infty(X)$ of X. The points of this space correspond to the "formal arcs" on X. Such an arc can be thought of as an infinitesimal curve centered at a point of X— that is, a choice of a point, together with a tangent direction at that point, a second order tangent direction, a

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third order tangent direction and so on. Thinking in this way, it is easy to believe that $J_\infty(X)$ is an infinite dimensional affine bundle over X.

(2) The measure. Motivic measure theory assigns to certain subsets of the arc space, called *cylinder sets,* a value from the motivic ring $\mathcal{M}$. Roughly speaking, a cylinder set is a subset in the infinite dimensional arc space $J_\infty(X)$ defined by polynomial equations in a *finite* set of the coordinates for $J_\infty(X)$.

The name cylinder set is descriptive, since such a set B may be thought of (at least locally) as the direct product of an infinite dimensional affine space with some variety B_m in a finite dimensional sub-affine space, given by algebraic conditions on the directions of i-th order tangent vectors, for $i = 0, \dots, m$, for some m. Now, choosing m to be minimal possible, the measure of the cylinder set B is defined as

$$\mu_X(B) = [B_m] \cdot \mathbb{L}^{-nm},$$

where $[B_m]$ is the class of B_m in $\mathcal{M}$, $\mathbb{L}$ is the class of the affine line in $\mathcal{M}$ (so that $\mathbb{L}^{-1}$ is its formal multiplicative inverse) and n is the dimension of X.

As a quick example which is, incidentally, crucial to the proof of Kontsevich's theorem, it is easy to check that the motivic measure of whole arc space $J_\infty(X)$ is

$$\mu(J_\infty(X)) = [X] \cdot \mathbb{L}^{-n \cdot 0} = [X],$$

the class of X in $\mathcal{M}$.

(3) Some interesting functions to integrate. Now fix any subvariety D of our smooth variety X. The subvariety D gives rise to an integer valued function F_D on the arc space $J_\infty(X)$, given, roughly speaking, by order of tangency of each arc along D. The value of $F_D(\gamma)$ may be zero (when γ misses D completely) or infinite (when γ is tangent to D to every order), or any finite number in between. The definition extends naturally to the case where D is a *divisor* on X, meaning a formal $\mathbb{Z}$-linear combination of codimension one subvarieties of X.

One example of an interesting measurable function on $J_\infty(X)$ is the function $\mathbb{L}^{-F_D}$. Because it takes discrete values, its integral over $J_\infty(X)$ is really just a large sum, which can be shown to converge in $\mathcal{M}$ (in a suitable topology).

(4) Kontsevich's Birational Transformation Rule. Let $f : X \longrightarrow Y$ be a proper birational morphism of smooth algebraic varieties. The Jacobian of this map is a function on X whose zero set defines a divisor $K_{X/Y}$ on X called the *relative canonical divisor* of the map. Kontsevich's birational transformation rule states that for an integrable function H on $J_\infty(Y)$ (taking values in $\mathcal{M}$),

$$\int_{J_\infty(Y)} H \, d\mu_Y = \int_{J_\infty(X)} (H \circ f^\infty) \, \mathbb{L}^{-F_{K_{X/Y}}} d\mu_X,$$

where f^∞ is the map from $J_\infty(X)$ to $J_\infty(Y)$ naturally induced by the map f from X to Y. Thus the "change of coordinates" factor is a function of the form $\mathbb{L}^{-F_{K_{X/Y}}}$ involving the usual Jacobian of the map f. This formula is non-trivial to prove, but essentially uses only fairly basic commutative algebra.

The proof of Kontsevich's Theorem about Calabi-Yau Manifolds. It is easy to grasp the main ideas of Kontsevich's proof of the opening conjecture. First let us first understand the conjecture more precisely.

A Calabi-Yau manifold is a smooth complex projective algebraic variety of dimension n admitting a nowhere vanishing holomorphic n-form—in other words, it has trivial canonical bundle.[1] Two complex manifolds are birationally equivalent if there are mutually inverse maps between them given locally by rational functions in local coordinates; these maps need not be defined on the closed sets where the denominators of the rational functions vanish. For any non-negative integers p and q, the Hodge number $h_{pq}(X)$ is the dimension of the space of closed (smooth) pq-forms on X modulo the space of exact pq-forms on X. Kontsevich's theorem states: *If X and Y are birationally equivalent smooth complex projective varieties, each of whose canonical bundle is trivial, then $h_{pq}(X) = h_{pq}(Y)$ for all $p, q \geq 0$.*

To prove this, Kontsevich considers the map from the set of all smooth projective varieties to the polynomial ring $\mathbb{Z}[u, v]$, defined by sending a variety X to the polynomial $\sum_{pq} h_{pq}(X)u^p v^q$. His idea is that (using some basic Hodge theory) this map factors through the motivic ring $\mathcal{M}$. He then shows that two birationally equivalent Calabi-Yau manifolds map to the same element in $\mathcal{M}$. Therefore, mapping further to $\mathbb{Z}[u, v]$, they must of course have the same Hodge numbers.

To see that two birationally equivalent Calabi-Yau manifolds map to the same class in $\mathcal{M}$, Kontsevich uses the fact that the measure of the whole arc space $J_\infty(X)$ is $[X]$. Thus

$$[X] = \int_{J_\infty(X)} d\mu_X,$$

and the point is to use the birational transformation rule to compute this integral in a different way.

Let Z be any smooth variety admitting a proper birational morphism $g : Z \longrightarrow X$ to a variety X. The birational transformation rule ensures that

$$[X] = \int_{J_\infty(X)} d\mu_X = \int_{J_\infty(Z)} \mathbb{L}^{-F_{K_{Z/X}}}.$$

Now simple geometric considerations ensure that, *when X has trivial canonical bundle,* the relative canonical divisor $K_{Z/X}$ depends only on Z. But now if X and Y are birationally equivalent, there exists a smooth variety Z admitting a proper birational morphism to both (a so-called *resolution of indeterminacies*). So if X and Y are birationally equivalent varieties with trivial canonical bundles, then the classes $[X]$ and $[Y]$ in $\mathcal{M}$ can both be computed as the same motivic integral! This means that X and Y have the same image in $\mathcal{M}$, and hence the same image in $\mathbb{Z}[u, v]$ and the same Hodge numbers.

Other applications of Motivic Integration. This beautiful proof of Kontsevich is only the beginning of a long story featuring many elegant applications of motivic integration to problems in algebraic and arithmetic geometry, number theory, representation theory and theoretical physics (including zeta functions, p-adic integration, MacKay correspondence, string theory, and mirror symmetry). In our seminar this year at MSRI, held as part of the Special Year in Commutative Algebra, we have focused on applications to higher dimensional birational geometry (such as characterizations of singularities and formulas for log canonical thresholds). Some of our notes and a collection of electronic literature can be found on our seminar web site: www.mabli.org/jet.html.

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[1]The precise definition of a Calabi-Yau manifold also requires certain cohomology groups to vanish, but this assumption is not necessary for Kontsevich's theorem.