GROUPS OF ORDER AUTOMORPHISMS OF CERTAIN
HOMOGENEOUS ORDERED SETS

David Eisenbud

1. INTRODUCTION

Call a chain (linearly ordered set) short if it contains a countable unbounded
subset, and homogeneous if all convex subsets without greatest or least elements
are isomorphic. The purpose of this paper is to investigate the algebraic structure
of the group $S(\Omega)$ of order automorphisms of a short homogeneous chain (abbrevi-
ated SHC) $\Omega$.

In Section 2 we show that the group structure of $S(\Omega)$ determines, up to duality,
the structure of $\Omega$ (the conditional completion of $\Omega$) and the lattice structure
of $S(\Omega)$. We give a partial solution to the problem of finding all SHC’s $\Omega$ with
the same group $S(\Omega)$. Our solution includes the result $S(\mathcal{R}) \not\cong S(\mathcal{A})$.

In Section 3 we calculate the automorphism groups of large subgroups of $S(\Omega)$. Our
result includes the theorem of J. T. Lloyd [5] that if $\Omega$ is conditionally complete, then
every automorphism of $S(\Omega)$ comes from conjugation by an order automor-
phism or anti-automorphism of $\Omega$.

The author is grateful to Otto H. Kegel and Peter M. Neumann for many enlighten-
ing discussions concerning this material.

Some notation: $S^\Omega$ is the full group of permutations of $\Omega$; $L(\Omega)$ (respectively,
$R(\Omega)$) is the subgroup of elements of $S(\Omega)$ whose support is bounded on the right (on
the left); and $N(\Omega) = R(\Omega) \cap L(\Omega)$. For unexplained terminology, see [7] and [1].

We note that not every SHC is a subset of $\mathcal{R}$ (the set of real numbers). See, for
example, [6].

2. GROUP STRUCTURE AND ORDER

The following is the fundamental tool of this paper.

THEOREM 1. If $\Omega$ is short, and all of its open intervals are isomorphic, then
$L(\Omega)$, $R(\Omega)$, and $N(\Omega)$ are the only proper normal subgroups of $S(\Omega)$; also, $N(\Omega)$ is
the only proper normal subgroup of $L(\Omega)$ or $R(\Omega)$, and $N(\Omega)$ is algebraically simple.

The difficult part of this, the simplicity of $N(\Omega)$, is due to G. Higman [2] (see
also [7, p. 25]). The rest of Theorem 1 is a consequence of [3, Theorem 6]. A
proof also appears in [5].

Note that if $\Omega$ is isomorphic to its order dual $\Omega^*$, then $L(\Omega) \cong R(\Omega)$ and all four
of the simple factors $S(\Omega)/L(\Omega)$, $S(\Omega)/R(\Omega)$, $R(\Omega)/N(\Omega)$, and $L(\Omega)/N(\Omega)$ are iso-
morphic. To complete the picture, we state without proof the following theorem.

THEOREM 2. Under the hypothesis of Theorem 1, $N(\Omega) \not\cong R(\Omega)/N(\Omega)$.

It is not to be hoped, even if $\Omega$ is an SHC, that the algebraic structure of $S(\Omega)$
will determine $\Omega$; for example, if $\Gamma$ is the set of irrational numbers and $\mathcal{A}$ is the

Received May 27, 1968.
set of rationals, then $S(\Gamma) \cong S(\mathcal{D})$. Proof: an element of $S(\Gamma)$ determines an element of $S(\mathcal{D})$, and the restriction to $\mathcal{D}$ of this permutation is an element of $S(\mathcal{D})$ [3, p. 407]. However, we shall show that in a certain sense this is the only way things can go wrong (see Corollary 3 to Theorem 4). We shall also prove that up to duality the lattice structure of $S(\Omega)$ is determined by the structure of $S(\Omega)$ as an abstract group.

First we establish some more terminology. If $\Omega$ is an SHC, we set

$$U(\Omega) = \{s \in S(\Omega) \mid s \text{ can be written uniquely as the product of commuting elements } s = lr = rl \text{ with } l \in L(\Omega), \ r \in R(\Omega)\}.$$ 

If $s \in S(\Omega)$ has a fixed block $\Lambda$, then the element $t \in S(\Omega)$ satisfying $t|\Omega - \Lambda = 1$ (the identity permutation of $\Omega - \Lambda$) and $t|\Lambda = s|\Lambda$ will be called the restriction of $s$ to $\Lambda$. We abbreviate minimal convex fixed block to MCFB, and we abbreviate support to supp. For purposes of exposition, we regard $\Omega$ as running from left to right.

**THEOREM 3.** If $\Omega$ is an SHC and $s \in S(\Omega)$, then $s \in U(\Omega)$ if and only if $s$ has exactly two nontrivial MCFB’s and at most one fixed point.

**Proof, $\Rightarrow.$** If $s$ had at least three nontrivial MCFB’s, then one of them would be bounded and we could write

$$s = (ln)r = r(ln) = l(nr) = (nr)l,$$

where $n \in N(\Omega)$ is the restriction of $s$ to the bounded fixed block, and $l \in L(\Omega)$ (respectively, $r \in R(\Omega)$) is the restriction of $s$ to the part of $\Omega$ to the left of the bounded fixed block (to the right of the bounded fixed block). Therefore $s \not\in U(\Omega)$, a contradiction.

Moreover, if $s \in U(\Omega)$ had two distinct fixed points, then because it can have at most two nontrivial MCFB’s, it would have to fix each point of some nontrivial bounded interval $[x, y]$. Then we could write

$$s = (lt^{-1})(tr) = (tr)(lt^{-1}) = lr = rl,$$

where $l$ is the restriction of $s$ to $(-\infty, x)$, $r$ is the restriction of $s$ to $(y, +\infty)$, and $t \neq 1$ is any element of $N(\Omega)$ with supp $t \subset [x, y]$. Hence $s \not\in U(\Omega)$, again a contradiction.

It remains to show that $s = rl \in U(\Omega)$ cannot have exactly one MCFB. Since $s|(-\infty, x) = 1|(-\infty, x)$ for any $x$ lying to the left of supp $r$, $l$ must have an MCFB $\Lambda \subset \Omega$ that is unbounded to the left. Since $lr l^{-1} = r$, supp $r$ is invariant under $l$ and $l^{-1}$; hence (supp $r$) $\cap \Lambda = \emptyset$. Consequently, $\Lambda$ is an MCFB of $s$.

$\Leftarrow$. Let $\Lambda_1$ and $\Lambda_2$ be the two MCFB’s of $s$, $\Lambda_1$ lying to the left of $\Lambda_2$. Let $l \in L(\Omega)$ (respectively, $r \in R(\Omega)$) be the restriction of $s$ to $\Lambda_1$ (to $\Lambda_2$). Then $s = lr = rl$. Suppose also that $s = l'r' = r'l'$, with $l' \in L(\Omega)$ and $r' \in R(\Omega)$. As in the first part of this proof, it can be shown that supp $l' \subset$ supp $l$, supp $r' \subset$ supp $r$, and therefore $l = l'$, $r = r'$.

**Remark.** To each element $lr$ of $U(\Omega)$ corresponds a nonextremal Dedekind cut $[D_1, D_2]$ with
GROUPS OF ORDER AUTOMORPHISMS OF ORDERED SETS

\[ D_1 = \text{supp } l, \quad D_2 = \text{the closure in } \Omega \text{ of supp } r. \]

Given a nonextremal Dedekind cut of \( \Omega \), one may always find many elements of \( U(\Omega) \) that correspond to it in this way.

**Definition.** If \( lr, l'r' \in U(\Omega) \), then

\[ lr \sim l'r' \iff 1 \text{ commutes with } r' \text{ and } l' \text{ commutes with } r. \]

Clearly,

\[ lr \sim l'r' \iff lr \text{ and } l'r' \text{ correspond to the same Dedekind cut.} \]

We write \( \{lr\} \) for the \( \sim \)-equivalence class of \( lr \).

**Definition.** If \( \{lr\}, \{l'r'\} \in U(\Omega)/\sim \), then

\[ \{lr\} \leq \{l'r'\} \iff 1 \text{ commutes with } r'. \]

It is now easy to prove the following theorem.

**Theorem 4.** If \( \Omega \) is an SHC, then \( U(\Omega)/\sim \cong \widetilde{\Omega} \), and the representation of \( S(\Omega) \) on \( \widetilde{\Omega} \) by extension of its action on \( \Omega \) is order-equivalent to its action on \( U(\Omega)/\sim \) by conjugation. In particular, if \( \Omega_1, \Omega_2 \) are both SHC's with \( S(\Omega_1) \cong S(\Omega_2) \), then either \( \widetilde{\Omega}_1 \cong \widetilde{\Omega}_2 \) or \( \widetilde{\Omega}_1 \cong \widetilde{\Omega}_2^* \).

*Note.* The confusion between \( \widetilde{\Omega}_2 \) and \( \widetilde{\Omega}_2^* \) arises as follows: given an abstract group that happens to be an \( S(\Omega) \) for some SHC \( \Omega \), our procedure for reconstructing \( \Omega \) through \( U/\sim \) begins with the choice of one of the two maximal normal subgroups to play the role of \( L(\Omega) \). If we choose the real \( L(\Omega) \), we get the isomorphism \( \widetilde{\Omega} \cong U/\sim \). If by accident we choose what is actually \( R(\Omega) \), we get the isomorphism \( U/\sim \cong \widetilde{\Omega}^* \).

The lattice structure of \( S(\Omega) \) can also be recovered from \( U(\Omega)/\sim \); if \( s, t \in S(\Omega) \), then

\[ s \leq t = \{u^s\} \leq \{u^t\} \quad \text{for all } u \in U(\Omega). \]

Note that \( U(\Omega) \) is normal in \( S(\Omega) \), and that we could as well have written \( \{u\}^s \leq \{u\}^t \).

**Corollary 1.** If \( \Omega_1 \) and \( \Omega_2 \) are SHC's with \( S(\Omega_1) \cong S(\Omega_2) \) as groups, then either \( S(\Omega_1) \cong S(\Omega_2) \) or \( S(\Omega_1) \cong S(\Omega_2)^* \) as lattice-groups.

**Corollary 2.** If \( \Omega \) is an SHC and \( \Omega \neq \overline{\Omega} \), then \( S(\Omega) \not\cong S(\overline{\Omega}) \).

*Proof.* \( S(\Omega) \) acts transitively on \( U(\Omega)/\sim \), but \( S(\Omega) \) is not transitive on \( U(\Omega)/\sim \) (one orbit consists of the classes of elements actually having fixed points, that is, the elements of \( U(\Omega) \) corresponding to the principal cuts of \( \Omega \)). In particular, \( S(\Omega) \not\cong S(\Omega') \) as groups (see also [3, p. 407]).

**Corollary 3.** If \( \Omega \) is an SHC, then each SHC \( \Omega' \) with \( S(\Omega) = S(\Omega') \) is order-isomorphic or anti-isomorphic to some conjugacy class of \( U(\Omega)/\sim \), and the group of order-automorphisms of each homogeneous conjugacy class is \( S(\Omega) \). In particular, the SHC's with \( S(\Omega) = S(\Omega') \) appear as disjoint, dense subsets of \( \overline{\Omega} \).
3. THE AUTOMORPHISMS OF LARGE SUBGROUPS OF \( S(\Omega) \)

**Convention.** If \( G \leq S^\Omega \), with \( \text{Cent}_{S^\Omega}(G) = \langle 1 \rangle \), and if \( s \in \text{Norm}_{S^\Omega}(G) \), we shall identify \( s \) with the automorphism of \( G \) given by conjugation by \( s \). Recall that if \( \Omega \) is any set with \( |\Omega| \neq 6 \) and \( A^\Omega \leq G \leq S^\Omega \) (\( A^\Omega \) denotes the alternating group on \( \Omega \)), then \( \text{Aut}(G) = \text{Norm}_{S^\Omega}(G) \) [7, Section 4]. The next theorem is an analogue of this for groups of order automorphisms. We regard \( S(\Omega) \) as a subgroup of \( S(\overline{\Omega}) \) in the obvious way.

**Theorem 5.** If \( \Omega \) is an SHC and \( N(\Omega) \leq G \leq S(\Omega) \), then

1. \( \text{Aut}(S(\Omega)) = \text{Norm}_{S(\Omega)}(S(\Omega)) \),
2. \( \text{Aut} G = \text{Norm}_{\text{Aut } S(\Omega)}(G) \).

**Proof of (1).** Identify the points of \( \overline{\Omega} \) with the classes in \( U(\Omega)/\sim \) (see Corollary 1). Then, for \( 1r \in U(\Omega) \),

\[
\text{Stab}_{S(\Omega)} \{1r\} = \{ y \in R(\Omega) \mid y \text{ commutes with } 1 \} \cdot \{ x \in L(\Omega) \mid x \text{ commutes with } r \}.
\]

This expression for the stabilizer is invariant under automorphisms of \( S(\Omega) \), and we conclude that the automorphisms of \( S(\Omega) \) permute the stabilizers of points of \( \overline{\Omega} \). By [7, Theorem 4.4], each automorphism is thus an element of \( S^\Omega \).

**Remark.** Holland has proved that if \( \Omega \) contains a nontrivial order-complete interval and \( S(\Omega) \) is transitive, then every lattice-automorphism of \( S(\Omega) \) is an inner automorphism [4, Theorem 8].

The proof of part 2 of Theorem 5 depends on properties of the topology \( \tau \) defined on \( S(\Omega) \) by taking as a sub-base for the open neighborhoods of \( s \in S(\Omega) \) the sets \( B_n(s) = s \text{ Cent}_{S(\Omega)}(n) \), where \( n \in N(\Omega) \). Convergence in this topology of a net \( \{ s_\alpha \}_{\alpha \in \Lambda} \subset S(\Omega) \) can be described in terms of \( \Omega \) by

\[
\lim s_\alpha = s \iff s_\alpha \mid \Lambda = s \mid \Lambda \text{ eventually for each bounded subset } \Lambda \text{ of } \Omega
\]

and every net \( \{ s_\alpha \}_{\alpha \in \Lambda} \subset S(\Omega) \).

**Lemma 1.**
1) The topology \( \tau \) is \( T_2 \).
2) A sequence in \( S(\Omega) \) converges if it is fundamental.
3) \( N(\Omega) \) is sequentially dense in \( S(\Omega) \).
4) Every automorphism of \( S(\Omega) \) is a homeomorphism of \( S(\Omega) \).
5) If \( \{ m_1 \} \) and \( \{ n_1 \} \) are convergent sequences in \( N(\Omega) \) such that \( \lim m_1 = \lim n_1 \in S(\Omega) \) and if \( \psi \in \text{Aut } N(\Omega) \), then \( \{ m_1^\psi \} \) and \( \{ n_1^\psi \} \) are convergent and \( \lim (m_1^\psi) = \lim (n_1^\psi) \).
6) If \( \{ m_1 \} \), \( \{ n_1 \} \), and \( \{ l_1 \} \) are convergent sequences in \( N(\Omega) \) and \( \psi \in \text{Aut } N(\Omega) \), then

\[
(\lim m_1)(\lim n_1) = \lim l_1 \implies (\lim m_1^\psi)(\lim n_1^\psi) = \lim l_1^\psi.
\]

The proof is straightforward, involving only the definition of \( \tau \) and the statement preceding the lemma.
LEMMA 2. \( \text{Aut } N(\Omega) = \text{Aut } S(\Omega) \).

Proof. Since \( N(\Omega) \) is a characteristic subgroup, it suffices to show that every element \( \psi \in \text{Aut } N(\Omega) \) extends uniquely to a \( \overline{\psi} \in \text{Aut } S(\Omega) \). Define \( \overline{\psi} \) as follows: for any \( s \in S(\Omega) \), pick a sequence \( \{ n_i \} \) in \( N(\Omega) \) with \( \lim n_i = s \). Set \( \overline{S\psi} = \lim n_i \psi \). Lemma 1 assures us that \( \overline{\psi} \) is well-defined and \( \overline{\psi} \in \text{Aut } S(\Omega) \).

Proof of (2). If \( \psi \in \text{Aut } G \), then \( \psi \) is determined by its restriction to \( N(\Omega) \), by Lemma 1 \( N(\Omega)\overline{\psi} = N(\Omega) \) because \( N(\Omega) \) is normal and simple and has trivial centralizer). By Lemma 2, \( \psi \) can be extended to be an element of \( \text{Aut } S(\Omega) \).

REFERENCES


University of Chicago
Chicago, Illinois 60637