JOURNAL OF ALGEBRA 47, 231–243 (1977)

What Annihilates a Module?*

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Communicated by the Editors

Received September 10, 1975

INTRODUCTION

On well-known method of approximating the annihilator of a finitely generated torsion module M over a commutative ring R is by means of fitting ideals. If

$$R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$$

is a free presentation of M, the first fitting ideal $F_1(M)$ is the ideal generated by the $n \times n$ minors of a matrix for φ. Writing ann M for the annihilator of M, we have

$$F_1(M) \subseteq \operatorname{ann} M,$$

and in fact the two ideals always have the same radical. If, more generally, we define the kth fitting ideal $F_k(M)$ to be the ideal of $(n-k+1) \times (n-k+1)$ minors of φ, we have

$$F_k(M) \subseteq \operatorname{ann} \bigwedge^k M, \tag{*}$$

where $\wedge^k$ denotes the kth exterior power, and again the two ideals have the same radical.

In this paper we are concerned with inequalities of this type, particularly those involving the annihilators of exterior and symmetric powers of M and annihilators of the cokernels of exterior and symmetric powers of φ, and in the question of when these inequalities can be replaced by equalities.

Recall that an ideal I in a noetherian ring R is said to have grade g (Bourbaki: $\operatorname{depth}_I R = g$) if I contains an R-sequence of length g, and that a module is said to have grade g if its ***annihilator*** has grade g. (Thus, the grade of an ideal $I \subset R$ is really the grade of R/I.) It is known [2] that the grade of the ideal of $l \times l$ minors of an $m \times n$ matrix cannot be greater than $(m-l+1)(n-l+1)$, and thus if

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* Both authors were partially supported during the preparation of this work by the NSF. The second author was partially supported by the Sloan Foundation.

ISSN 0021-8693

M is a module having n generators with m relations, grade $F_k(M) \leqslant k(m - n + k)$. Since ann M has the same radical as $F_1(M)$, we have

$$\text{grade ann } M \leqslant m - n + 1.$$

Our main theorem (Theorem 3.2) states that if this grade is achieved, then the annihilators of M and a number of related modules are equal to $F_1(M)$. Writing S_p for the pth symmetric power functor, we can state a special case of Theorem 3.2 as follows:

THEOREM. *Let R be a noetherian ring, and let*

$$R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0$$

be a free presentation of an R-module M, with $m > n$. Suppose that grade $M = m - n + 1$. Then

$$\text{ann } M = F_1(M).$$

More generally,

$$\text{ann } S_r(M) = \text{ann}\left(\text{coker} \bigwedge^q \varphi\right) = \text{ann}(\text{coker}(S_p\varphi)) = F_1(M)$$

for every p, q and r with $1 \leqslant p$ and $1 \leqslant q \leqslant n$, and $1 \leqslant r \leqslant m - n$.

If $m = n$ and det φ is a nonzero divisor, then

$$\text{ann } M = (F_1(M) : F_2(M)).$$

From our inequalities for the fitting ideals, we obtain another case in which $F_1(M)$ is the annihilator of coker $\wedge^q \varphi$ for each q: namely, the case in which $F_2(M)$ contains a nonzero divisor modulo $F_1(M)$; this generalizes a theorem of Eisenreich [3, Appendix].

The proofs of our generalizations of the inequalities (*) and the theorem of Eisenreich just mentioned involve only very simple multilinear algebra (although we do use the fact that the elements of F^* act as derivations of degree -1 on the graded algebra $\wedge F$); this is all done in Section 1. The proof of our main theorem, however, involves the particular structure of the free resolutions for coker $(\wedge^q \varphi)$ and coker$(S_p\varphi)$ constructed in [1]. These resolutions are built up from certain "multilinear functors" $L_p{}^q$ which represent a common generalization of both exterior and symmetric powers. (J. Towber has pointed out to us that the $L_p{}^q$ are obtainable as certain irreducible representations of symmetric groups; though we give definitions of $L_p{}^q$ which are satisfactory only for free modules, the definition can be extended using his ideas.) Section 2 contains a description of the $L_p{}^q$ and the resolutions $\mathbf{L}_p{}^q(\varphi)$ that we need for the proof of our main theorem; further details of the construction, plus a survey of the necessary multilinear algebra, can be found in [1].

1. INEQUALITIES ON ANNIHILATORS

Let $\varphi\colon F \to G$ be a map of free R-modules, with cokernel M. For any integers $s, t \geqslant 0$, φ induces a map

$$\varphi_{s,t}\colon \bigwedge^{s} F \otimes \bigwedge^{t} G \to \bigwedge^{s+t} G$$

by

$$\alpha \otimes \beta \mapsto \bigwedge^{s} \varphi(\alpha) \wedge \beta.$$

DEFINITION. $I(s, t) = \operatorname{ann}(\operatorname{coker} \varphi_{s,t})$.

It is easy to see (using Lemma 1.1 and Theorem 1.2, part 2 below) that, if G has rank n, the ideals $I(n - s, t)$ depend on M, s, and t, but not on the presentation chosen. Also, the ideal $I(n - q + 1, q - 1)$ is nothing but the ideal of $(n - q + 1) \times (n - q + 1)$ minors of φ, and thus

$$F_q(M) = I(n - q + 1, q - 1).$$

Moreover, the annihilator of $\wedge^q M$ is given by

$$\operatorname{ann} \bigwedge^{q} M = I(1, q - 1);$$

this follows from part a of the following well-known lemma.

LEMMA 1.1. *With the above notation,*

(a) $\bigwedge^{q} M = \operatorname{coker} \varphi_{1,q-1}$

(b) *image* $\varphi_{s,t} \supseteq$ *image* $\varphi_{s+1,t-1}$ *for all* s, t.

Proof. It follows from the right exactness of the exterior algebra functor that

$$\bigwedge^{q} M = \operatorname{coker}\left(\sum_{s \geqslant 1} \bigwedge^{s} F \otimes \bigwedge^{q-s} G \to \bigwedge^{q} G\right);$$

Thus part a follows from part b. Part b may be verified directly, or from the diagram

$$\begin{tikzcd}
\bigwedge^{s} F \otimes F \otimes \bigwedge^{t} G \arrow[r, "1\otimes\varphi_{1,t}"] \arrow[d, "m\otimes 1"'] & \bigwedge^{s} F \otimes \bigwedge^{t+1} G \arrow[d, "\varphi_{s,t+1}"] \\
\bigwedge^{s+1} F \otimes \bigwedge^{t} G \arrow[r] & \bigwedge^{s+t+1} G
\end{tikzcd}$$

where the map m is the multiplication in the exterior algebra; the commutativity of the diagram follows from the associativity of the exterior algebra. ☐

Thus the following result gives information about the relationships between the fitting ideals of M and the annihilators of the exterior powers of M:

THEOREM 1.2. *With notation as above, we have*:

(1) $I(s, t) \subseteq I(s, t + 1)$.

(2) *If* $s + t \leqslant$ *rank* G, *then*

$$I(s - 1, t) \subseteq I(s, t).$$

(3) *For any* s', t', *set* $t'' = \max(t, t' - s)$. *Then*

$$I(s, t)\, I(s', t') \subseteq I(s + s', t'').$$

As a first consequence we have:

COROLLARY 1.3. *Let M be module which can be generated by n elements. Then*

$$\left(\operatorname{ann} \bigwedge^{s} M\right)^{n-s+1} \subseteq F_s(M) \subseteq \operatorname{ann} \bigwedge^{s} M.$$

This result can be improved under various circumstances, as in the following corollary, which is a generalization of the result of Eisenreich quoted in the introduction.

COROLLARY 1.4. *Let M be a finitely generated module*; *then* $F_s(M) \subseteq \operatorname{ann} \wedge^s M \subseteq (F_s(M) : F_{s+1}(M))$. *In particular, if $F_{s+1}(M)$ contains a nonzero divisor modulo $F_s(M)$, then*

$$F_s(M) = \operatorname{ann} \bigwedge^{s} M.$$

Proof of Corollary 1.3. Recall that $F_s(M) = I(n - s + 1, s - 1)$, and that $\wedge^s M = \operatorname{coker} \varphi_{1,s-1}$. The second inequality of the corollary now follows at once from part 2 of the theorem. For the first inequality, we take $s_1 = 1$, $s_2 = n - s$, $t_1 = t_2 = s - 1$ in part 3 of the theorem, obtaining

$$I(1, s - 1)\, I(n - s, s - 1) \subset I(n - s + 1, s - 1).$$

The result now follows by iteration.

Proof of Corollary 1.4. For the first statement it is enough to show $\operatorname{ann} \wedge^s M \subseteq F_s$. Taking $s_1 = n - s$, $t_1 = s - 1$, $s_2 = 1$, $t_2 = s$ in part 3 of the theorem, we obtain

$$I(1, s - 1)\, I(n - s, s) \subseteq I(n - s + 1, s - 1),$$

or,

$$\operatorname{ann}\left(\bigwedge^{s} M\right) F_{s+1} \subseteq F_s\,;$$

so if F_{s+1} contains a nonzero divisor modulo F_s, ann $\Lambda^s M \subseteq F_s$. □

Proof of Theorem 1.2. Part 1 follows from part 3, with $s_1 = s$, $t_1 = t$, $s_2 = 0$, $t_2 = t + s + 1$, and the observation that $I(0, t_2) = R$ for any t_2. Thus it suffices to prove parts 2 and 3.

Part 2. Take $g \in \Lambda^{s+t-1} G$, $r \in I(s, t)$. We wish to show that rg is in the image of $\varphi_{s-1,t} : \Lambda^{s-1} F \otimes \Lambda^t G \to \Lambda^{s+s-1} G$. Since G is free of rank n and $s + t \leqslant n$, there is an element $g' \in \Lambda^{s+t} G$ and an element $\gamma \in G^*$ such that $g = \gamma(g')$. But rg' *is* in the image of $\varphi_{s,t}$, so we may write

$$rg' = \sum \bigwedge^{s} \varphi(f_i) \wedge g_i' \qquad \text{with} \quad f_i \in \bigwedge^{s} F,$$
$$g_i' \in \bigwedge^{t} G.$$

Applying γ, we get

$$\begin{aligned} rg = \gamma(rg') &= \gamma\left(\sum \bigwedge^{s} \varphi(f_i) \wedge g_i'\right),\\ &= \sum \gamma\left(\bigwedge^{s} \varphi(f_i) \wedge g_i' + \sum \bigwedge^{s} \varphi(f_i) \wedge \gamma(g_i')\right), \qquad (*) \end{aligned}$$

since γ acts on $\Lambda\, G$ as a derivation.

But $\gamma(\Lambda^s \varphi(f_i)) = \Lambda^{s-1} \varphi(\varphi^*(\gamma)(f_i))$, so the first term in (*) is in the image of $\varphi_{s-1,t}$. The other term in (*) is clearly in the image of $\varphi_{s,t-1}$. By Lemma 1.1,

$$\text{Image } \varphi_{s,t-1} \subseteq \text{Image } \varphi_{s-1,t}\,,$$

so all of (*) is in Image $\varphi_{s-1,t}$ as desired.

Part 3. Let $u \in I(s_1, t_1)$, $v \in I(s_2, t_2)$. Then if $t = \max(t_1, t_2 - s_1)$, and $g \in \Lambda^{s_1+s_2+t} G$, we wish to show that uvg is in the image of $\varphi_{s_1+s_2,t}$. By linearity we may assume that g is a product of elements of degree 1, and since $s_1 + s_2 + t \geqslant s_2 + t_2$, we may write $g = hk$ with degree $h = s_2 + t_2$. By hypothesis, $vh = \sum \Lambda^{s_2} \varphi(f_i) \wedge h_i$, with $f_i \in \Lambda^{s_2} F$, $h_i \in \Lambda^{t_2} G$, and h_i is a product of elements of degree 1. Since the degree of $h_i k$ is $s_1 + s_2 + t - s_2 = s_1 + t \geqslant s_1 + t_1$, we may write $h_i k = l_i m_i$, where degree $l_i = s_1 + t_1$, and, again by the hypothesis,

$$ul_i = \sum \bigwedge^{s_1} \varphi(f'_{ij}) \wedge l_{ij}\,,$$

for some $f'_{ij} \in \Lambda^{s_1} F$, $l_{ij} \in \Lambda^{t_1} G$.

Thus

$$uvg = \sum_{ij} \bigwedge^{s_2} \varphi(f_i) \wedge \bigwedge^{s_1} \varphi(f'_{ij}) \wedge l_{ij} \wedge m_i$$

$$= \sum_{ij} \bigwedge^{s_1+s_2} \varphi(f_i \wedge f'_{ij}) \wedge (l_{ij} \wedge m_i) \in \text{Image } \varphi_{s_1+s_2,t}\,,$$

as required. □

A theorem similar to Theorem 1.2 can be worked out for the symmetric algebra, although the results differ in that all the annihilators of symmetric powers of a module have the same radical. We need only the following easy case, which will be used in Section 3.

PROPOSITION 1.5. *For any R-module M, and any $p \geqslant 1$,*

$$\text{ann } S_{p+1}(M) \supseteq \text{ann } S_p(M).$$

Proof. If G is a free module, and

$$\alpha\colon G \twoheadrightarrow M$$

is an epimorphism, then we have epimorphisms

$$G \otimes S_pM \twoheadrightarrow M \otimes S_pM \twoheadrightarrow S_{p+1}M,$$

where the last map is the multiplication map in $S(M)$. The inequality on annihilators follows. □

2. SOME MULTILINEAR FUNCTORS AND GENERIC FREE RESOLUTIONS

In [1] we defined certain functors $L_p{}^q$ on free R-modules, and used them to construct generic free resolutions. We now review the part of those results which we need for the proof of our main theorem. Details may be found in [1, Sects. 2, 3, and 4].

Suppose F is a finitely generated free R-module. Then the identity map $1\colon F \to F$ corresponds to an element

$$c \in F \otimes F^* = S_1(F) \otimes \bigwedge^1 F^* \subset S(F) \otimes \bigwedge F^*,$$

the tensor product of a symmetric algebra and an exterior algebra. Since $\wedge F$ is a $\wedge F^*$-module, multiplication by c induces, for every k, l, a map

$$\partial_k{}^l\colon S_{k-1}F \otimes \bigwedge^l F \to S_kF \otimes \bigwedge^{l-1} F.$$

DEFINITION. $L_p{}^qF = \ker \quad S_pF \otimes \wedge^{q-1} F \xrightarrow{\partial_{p+1}^{q-1}} S_{p+1}F \otimes \wedge^{q-2} F$. The naturality of this definition clearly makes $L_p{}^qF$ a functor of F. To interpret this definition, the following two propositions from [1] will be all we need:

PROPOSITION 2.1. (a) *$L_p{}^1F = S_pF$ for all $p \geqslant 0$.*

(b) *$L_1{}^qF = \wedge^q F$ for all $q > 0$.*

(c) *If rank $F = n$, then*

$$L_p{}^nF \cong S_{p-1}F \otimes \bigwedge^n F.$$

All these isomorphisms are natural in F.

If $\varphi: F \rightarrow G$ is a map of free modules, we can ask about the annihilators of the cokernels of $L_p{}^q\varphi$:

PROPOSITION 2.2. *If* coker $\varphi = M$, *we have*

$$(\operatorname{ann} M)^{p+q-1} \subseteq \operatorname{ann}(\operatorname{coker} L_p{}^q\varphi) \subseteq \operatorname{ann} M$$

for every p, q with $1 \leqslant p$, $1 \leqslant q \leqslant n$.

Because SF and $\wedge F$ are (graded) commutative algebras, ∂, which is multiplication by c, is a map of $SF \otimes \wedge F^*$-modules. Thus $LF = \sum_{p,q} L_p{}^qF$ is an $SF \otimes \wedge F^*$-module.

To any map $\varphi: F \rightarrow G$ there corresponds an element $c_\varphi \in F^* \otimes G \subset \wedge F^* \otimes S(G)$. If we now use the $\wedge F^*$-module structure on LF and the SG-module structure on $(LG)^* = \sum \operatorname{Hom}_R(L_p{}^qG, R)$, multiplication by c_φ induces a map

$$d: L_k{}^lF \otimes (L_r{}^sG)^* \rightarrow L_k^{l-1}F \otimes (L_{r-1}^sG)^*.$$

Also, the $\wedge F^*$-module structure on LF together with the ring homomorphism $\wedge G^* \rightarrow \wedge F^*$ induced by φ^* allows us to define a map

$$d_1: L_k{}^lF \otimes L_1{}^sG^* \cong L_k{}^lF \otimes \bigwedge^s G^* \rightarrow L_k^{l-s}F.$$

Assuming now that rank $F = m$ and rank $G = n$, we construct a complex:

$$\mathbf{L}_p{}^q(\varphi){:}\ 0 \longrightarrow L_p{}^mF \otimes L_{m-n}^{n-q+1}G^* \xrightarrow{d} L_p^{m-1}F \otimes L_{m-n-1}^{n-q+1}G^* \xrightarrow{d} \cdots$$

$$\xrightarrow{d} L_p^{n+1}F \otimes L_1^{n-q+1}G^* \xrightarrow{d_1} L_p{}^qF \xrightarrow{L_p{}^q\varphi} L_p{}^qG.$$

Recall that if R is a noetherian ring, an R-module M is said to be *perfect of grade g* if ann M contains an R-sequence of length g, and the projective dimension of M is g. We have:

THEOREM 2.3. *Suppose that*

$$\varphi: F \to G$$

is a map of free R-modules, with rank $F = m \geqslant$ rank $G = n$. *Further, suppose that the ideal* $F_1(\text{coker } \varphi)$ *of* $n \times n$ *minors of* φ *has grade* $m - n + 1$. *Then for each* p, q *with* $1 \leqslant p$, $1 \leqslant q \leqslant n$, *the complex* $\mathbf{L}_p{}^q(\varphi)$ *is exact, so that the projective dimension of* $\text{coker}(L_p{}^q(\varphi))$ *is* $m - n + 1$.

Combining this with Proposition 2.2, we get

COROLLARY 2.4. *Under the hypothesis of Theorem* 2.3, $\text{coker}(L_p{}^q\varphi)$ *is a perfect module of grade* $m - n + 1$.

3. THE MAIN THEOREM

We now come to the main result of this paper:

THEOREM 3.1. *Suppose that R is a noetherian ring, and M is an R-module with presentation*

$$R^m \xrightarrow{\varphi} R^n \longrightarrow M \longrightarrow 0,$$

satisfying grade $F_1(M) = m - n + 1$ (*the maximum possible value*). *Then*:

(1) *If* $m > n$ *then for all* p, q *with* $1 \leqslant q \leqslant n$, $1 \leqslant p$, *we have*

$$\text{ann}(\text{coker } L_p{}^q\varphi) = F_1(M).$$

(2) *If* $1 \leqslant p \leqslant m - n$, *then* ann $S_p(\text{coker } \varphi) = F_1M$.

(3) *If* $m = n$, *then*

$$\text{ann}\left(\text{coker} \bigwedge^p \varphi\right) = (F_1(M): F_{p+1}M).$$

Here $(F_1(M): F_{p+1}(M))$ *denotes as usual the "residual quotient," that is*, $\{r \in R \mid rF_{p+1}(M) \subseteq F_1(M)\}$.

Remark. Even under the hypothesis of Theorem 3.1, we may have ann $S_{m-n+1}(M) \supsetneqq$ ann $M = F_1(M)$. For example, let $R = k[x, y]$, and let φ: $R^3 \to R^2$ be the map whose matrix is

$$\varphi = \begin{pmatrix} x_1 & x_2 & 0 \\ 0 & x_1 & x_2 \end{pmatrix}.$$

Set $M = \operatorname{coker} \varphi$. We have $\operatorname{ann} M = F_1(M) = (x_1, x_2)^2$, but $\operatorname{ann} S_2(M) = (x_1, x_2)$. For we can choose generators e_1, e_2 of M, with

$$x_1e_1 = x_2e_2 = 0$$

$$x_2e_1 = -x_1e_2 .$$

But then $x_1e_1{}^2 = 0 = x_1e_1e_2$ is clear, and

$$x_1e_2{}^2 = -x_1e_1e_2 = 0$$

as well.

Proof of Theorem 3.1. We first dispose of part 3, which is elementary. In this case $F_1(M) = (\det \varphi)$, and the hypothesis is simply that $\det \varphi$ is a nonzerodivisor in R. We now prove that $\operatorname{ann}(\operatorname{coker} \wedge^p \varphi) = ((\det \varphi) : F_{p+1}(M))$ by proving the two inequalities separately:

(a) $\operatorname{ann}(\operatorname{coker} \wedge^p \varphi) \subseteq ((\det \varphi) : F_{p+1}(M))$. Let $r \in \operatorname{ann}(\operatorname{coker} \wedge^p \varphi)$. Then multiplication by r is homotopic to 0 on the complex

$$\begin{tikzcd} 0 \arrow[r] & \bigwedge^{p} R \arrow[r, "\wedge^p\varphi"] & \bigwedge^{p} R^n \arrow[l, bend left, "a"] \arrow[r] & 0. \end{tikzcd}$$

Thus there is a map a as pictured above such that $a(\wedge^p \varphi) = (\wedge^p \varphi)\, a = r \cdot 1$, where 1 is the identity map on $\wedge^p R^n$.

Now using the canonical isomorphism $\alpha : \wedge^p R^n \to \wedge^{n-p} R^{n*}$, we may factor $(\det \varphi) \cdot 1$ as follows:

$$\begin{tikzcd} \bigwedge^{p} R^n \arrow[r, "\alpha"] \arrow[rrrr, bend right, "(\det \varphi) \cdot 1"'] & \bigwedge^{n-p} R^{n*} \arrow[r, "\wedge^{n-p}\varphi^*"] & \bigwedge^{n-p} R^{n*} \arrow[r, "\alpha^{-1}"] & \bigwedge^{p} R^n \arrow[r, "\wedge^p\varphi"] & \bigwedge^{p} R^n \end{tikzcd}$$

Thus

$$\text{(i)} \quad r \cdot 1(\alpha^{-1} \wedge^{n-p} \varphi^* \alpha) = a \wedge^p \varphi\alpha^{-1} \wedge^{n-p} \varphi^*\alpha = a(\det \varphi \cdot 1).$$

Since the entries of a matrix for $\alpha^{-1} \wedge^{n-p} \varphi^*\alpha$ generate the ideal $F_{p+1}(M)$, the entries of the left-hand side of (i) generate $rF_{p+1}(M)$. But the entries of the right-hand side of (i) are clearly in the ideal $(\det \varphi)$. This shows that $r \in ((\det \varphi) : F_{p+1})$.

(b) $(\det(\varphi) : F_{p+1}(M)) \subseteq \operatorname{ann}(\operatorname{coker} \wedge^p \varphi)$.

Suppose $rF_{p+1}(M) \subset (\det \varphi)$. Then factorization

$$\begin{tikzcd} \bigwedge^{p} R^n \arrow[r, "\wedge^p\varphi"] \arrow[rrrr, bend right, "(\det \varphi) \cdot 1"'] & \bigwedge^{p} R^n \arrow[r, "\alpha"] & \bigwedge^{n-p} R^{n*} \arrow[r, "\wedge^{n-p}\varphi^*"] & \bigwedge^{n-p} R^{n*} \arrow[r, "\alpha^{-1}"] & \bigwedge^{p} R^n \end{tikzcd}$$

shows that a map $a\colon \bigwedge^p R^n \to \bigwedge^p R^n$ can be constructed to make the following diagram commute:

$$\begin{tikzcd}
\bigwedge^p R^n \arrow[dr, "\bigwedge^p \varphi"] \arrow[ddrr, bend left, "(\det \varphi)\cdot 1"] & & \\
& \bigwedge^p R^n \arrow[dr, "\alpha^{-1} \bigwedge^{n-p} \varphi^* \alpha"] & \\
& & \bigwedge^p R^n. \\
& \bigwedge^p R^n \arrow[ur, "\alpha^{-1} \bigwedge^{n-p} \varphi^* \alpha"'] & \\
\bigwedge^p R^n \arrow[ur, "r\cdot 1"] \arrow[uuuu, dashed, "a"] & &
\end{tikzcd}$$

But since $\det \varphi$ is a nonzero divisor, $\alpha^{-1} \bigwedge^{n-p} \varphi^* \alpha$ is a monomorphism, so

$$\begin{tikzcd}
\bigwedge^p R^n \arrow[dr, "\bigwedge^p \varphi"] & \\
& \bigwedge^p R^n \\
\bigwedge^p R^n \arrow[ur, "r\cdot 1"'] \arrow[uu, "a"] &
\end{tikzcd}$$

commutes, which shows that r annihilates coker $\bigwedge^p \varphi$. This concludes the proof of the third part of the theorem.

To prove parts 1 and 2 of the theorem, we will employ the following simple idea:

LEMMA 3.2. *Suppose M is a perfect module of grade g. Then* ann M = ann $\operatorname{Ext}^g(M, R)$.

Proof. Clearly ann $M \subseteq$ ann $\operatorname{Ext}^g(M, R)$. But since M has grade g,

$\mathrm{Ext}^k(M, R) = 0$ for $0 \leqslant k \leqslant g$, so the dual of a free resolution for M will be a free resolution for $\mathrm{Ext}^g(M, R)$, and we have

$$\mathrm{Ext}^g(\mathrm{Ext}^g(M, R), R) = M.$$

Thus $\mathrm{ann}\ \mathrm{Ext}^g(M, R) \subseteq \mathrm{ann}\ M$ as well. □

By Theorem 2.1, we may apply this lemma to the modules $\mathrm{coker}(L_p{}^q\varphi)$ which are perfect of grade $m - n + 1$ under the hypothesis of our theorem. The module $\mathrm{Ext}^{m-n+1}(\mathrm{coker}\, L_p{}^q\varphi, R)$ is the cokernel of the dual of the last differential in the free resolution of $\mathrm{coker}\, L_p{}^q\varphi$. If $\varphi\colon R^m = F \to R^n = G$, then this dual map is

$$\text{(ii)}\quad d^*_{m-n+1}\colon L^{n-q+1}_{m-n-1}G \otimes L_p^{m-1}F^* \to L^{n-q+1}_{m-n}G \otimes L_p{}^mF^*.$$

Applying the definitions of the various modules involved and of the map, we obtain a commutative diagram

$$\begin{tikzcd}
& L^{n-q+1}_{m-n-1}G \otimes L_p^{m-1}F^* \arrow[r, "d^*_{m-n+1}"] & L^{n-q+1}_{m-n}G \otimes L_p{}^mF^* \\
\text{(iii)} & L^{n-q+1}_{m-n-1}G \otimes \bigwedge^{m-1} F^* \otimes S_{p-1}F^* \arrow[r] \arrow[u, two heads] & L^{n-q+1}_{m-n}G \otimes \bigwedge^{m} F^* \otimes S_{p-1}F^* \arrow[u, equal, "\wr"'] \\
& L^{n-q+1}_{m-n-1}G \otimes F \otimes S_{p-1}F^* \arrow[r, "m\otimes 1"'] \arrow[u, dash, "\wr"'] & L^{n-q+1}_{m-n}G \otimes S_{p-1}F^*, \arrow[u, equal, "\wr"']
\end{tikzcd}$$

where $m\colon L^{n-q+1}_{m-n-1}\, G \otimes F \to L^{n-q+1}_{m-n}\, G$ is induced by $\varphi\colon F \to G$ and the module structure map $L^{n-q+1}_{m-n-1}\, G \otimes G \to L^{n-q+1}_{m-n}\, G$. But this m is clearly d^*_{m-n+1} in case $p = 1$, that is, in the resolution of $\mathrm{coker}(L_1{}^q\varphi) = \mathrm{coker}(\wedge^q \varphi)$. Thus

$$\mathrm{Ext}^{m-n+1}(\mathrm{coker}\, L_p{}^q\varphi, R) = S_{p-1}F^* \otimes \mathrm{Ext}^{m-n+1}\left(\mathrm{coker} \bigwedge^q \varphi, R\right),$$

which has the same annihilators as $\mathrm{coker} \wedge^q \varphi$, by the lemma. We are thus reduced to the case $p = 1$. Of course $F_1(M)$ is by definition $\mathrm{ann}(\mathrm{coker} \wedge^n \varphi)$, and by part 2 of Theorem 1.2,

$$\begin{aligned}\mathrm{ann}\ M &= \mathrm{ann}\left(\mathrm{coker} \bigwedge^1 \varphi\right) \supseteq \mathrm{ann}\left(\mathrm{coker} \bigwedge^2 \varphi\right) \supseteq \cdots \supseteq \mathrm{ann}\left(\mathrm{coker} \bigwedge^n \varphi\right) \\ &= F_1(M).\end{aligned}$$

Thus it suffices to show that $\mathrm{ann}\ M \subseteq F_1(M)$. Using ii and iii, and the identifications of $L_p{}^q$ given in Section 2, and the duality in the exterior algebra, we compute:

$$\begin{aligned}
&\mathrm{Ext}^{m-n+1}(M, R)\\
&\quad = \operatorname{coker}(L^n_{m-n-1}G \otimes L_1^{m-1}F^* \to L^n_{m-n}G \otimes L_1{}^mF^*)\\
&\quad = \operatorname{coker}\left(\overset{n}{\bigwedge} G \otimes S_{m-n-2}G \otimes \overset{m-1}{\bigwedge} F^* \to \overset{n}{\bigwedge} G \otimes S_{m-n-1}G \otimes \overset{m}{\bigwedge} F^*\right)\\
&\quad = \operatorname{coker}(S_{m-n-2}G \otimes F \xrightarrow[m]{} S_{m-n-1}G)\\
&\quad = S_{m-n-1}(\operatorname{coker} \varphi),
\end{aligned}$$

while

$$\begin{aligned}
&\mathrm{Ext}\left(\operatorname{coker} \overset{q}{\bigwedge} \varphi, R\right)\\
&\quad = \operatorname{coker}(L^1_{m-n-1}G \otimes L_1^{m-1}F^* \to L^1_{m-n}G \otimes L_1{}^mF^*)\\
&\quad = \operatorname{coker}(S_{m-n-1}G \otimes F \to S_{m-n}G)\\
&\quad = S_{m-n}(\operatorname{coker} \varphi).
\end{aligned}$$

Thus

$$\begin{aligned}
\text{(iv)}\quad \operatorname{ann} M &= \operatorname{ann} S_{m-n-1}(M)\\
F_1(M) &= \operatorname{ann} S_{m-n}(M).
\end{aligned}$$

But by Proposition 1.5,

$$\operatorname{ann} S_{m-n-1}(M) \subseteq \operatorname{ann} S_{m-n}(M),$$

which gives the required inequality, proving part one of the theorem.

For part 2 of the theorem, note that by Proposition 1.5,

$$\operatorname{ann} M = \operatorname{ann}(S_1M) \supseteq \operatorname{ann}(S_2M) \supseteq \cdots \supseteq \operatorname{ann} S_{m-n}(M).$$

But by iv, and part 1 of Theorem 3.1, ann $M = \operatorname{ann} S_{m-n}(M)$, so all the ideals in this sequence are actually equal. ☐

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