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A Nullstellensatz with Nilpotents and Zariski's Main Lemma on Holomorphic Functions

DAVID EISENBUD AND MELVIN HOCHSTER*

Department of Mathematics, Brandeis University, Waltham, Massachusetts 02254
and
Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

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The classical Nullstellensatz asserts that a reduced affine variety is known by its closed points; algebraically, a prime ideal in an affine ring is the intersection of the maximal ideals containing it. A leading special case of our theorem says that any affine scheme can be distinguished from its subschemes by its closed points with a bounded index ot nilpotency; algebraically, an ideal I in an affine ring A may be written as

$$I = \bigcap_{\mathfrak{m} \in \mathcal{N}} (\mathfrak{m}^e + I), \qquad (*)$$

where $\mathcal{N}$ is the set of maximal ideals containing I, and e is an integer depending on the degree of nilpotency of A/I.

Our theorem might also be thought of as a sharpening of Zariski's Main Lemma on holomorphic functions [4]. Roughly speaking, this lemma asserts that if a regular function f on an irreducible affine variety V vanishes to order e at each of a dense set $\mathcal{N}$ of closed points of V, then it vanishes to order e at the generic point; that is, if P is the prime ideal in $k[x_1, ..., x_n]$ defining V, then

$$\bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^e = P^{(e)}, \qquad (**)$$

the eth symbolic power of P, where the intersection is taken over a dense set of maximal ideals $\mathfrak{m}$ of $k[x_1, ..., x_n]$ containing P. Of course this implies that, if I is a P-primary ideal containing $P^{(e)}$, then

$$I \supset \bigcap_{\mathfrak{m} \in \mathcal{N}} \mathfrak{m}^e;$$

(*), above, is a sharpening that includes (**).

Our proof is related to Zariski's but is simpler than his.

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Throughout this paper, all rings will be commutative and Noetherian, with identity. If R is a ring and P is a prime ideal of R, a finitely generated module M is said to be *P-coprimary* if P is the only associated prime of M [2]. The *e*th *symbolic power* $P^{(e)}$ *of* P is by definition the inverse image of $P_P{}^e$ in R ($P_P{}^e \cap R$ if R is a domain), which is the P-primary component of P^e.

Results

THEOREM. *Let R be a ring, and let P be a prime ideal of R. Let $\mathcal{N}$ be a set of maximal ideals m containing P such that R_m/P_m is a regular local ring, and such that*

$$\bigcap_{m \in \mathcal{N}} m = P.$$

If M is a finitely generated P-coprimary module annihilated by P^e, then

$$\bigcap_{m \in \mathcal{N}} m^e M = 0.$$

(Note: Since M is P-coprimary, P^e annihilates M if and only if $P^{(e)}$ does.)

COROLLARY 1 (Zariski's Main Lemma on Holomorphic Functions [4]). *With R, P, and $\mathcal{N}$ as above we have*

$$P^{(e)} \supseteq \bigcap_{m \in \mathcal{N}} m^e \supseteq \bigcap_{m \in \max \operatorname{spec} R, m \supset P} m^e.$$

If R is regular, the inclusions may be replaced by equalities.

Proof. The first statement, which is the original "Lemma," follows from our theorem with $M = R/P^{(e)}$. For the second statement, it suffices to prove $m^e \supset P^{(e)}$ for any $m \supset P$; this is the content of a theorem of Nagata [3, p. 143] and Zariski (see [1, Theorem 1]). The next corollary answers a question of B. Wehrfritz which originally motivated this study.

COROLLARY 2. *Let A be a ring finitely generated over a field or over the integers, and let M be a finitely generated A-module. For sufficiently large e, we have*

$$\bigcap_{m \in \max \operatorname{spec} A} m^e M = 0.$$

(In fact, if $0 = \bigcap M_i$ is a primary decomposition of $0 \subset M$, with M/M_i P_i-coprimary, and $P_i^{e_i}(M/M_i) = 0$, then we may take e to be the maximum of the e_i).

Proof. By [2, Ch. 12 and 13], the regular locus of any domain A/P, finitely generated over the integers or a field, is open and therefore dense. Thus we may apply our Theorem to the coprimary modules M/M_i, with the desired result. ∎

Remarks. (1) Clearly, it suffices in Corollary 2 that A be an excellent Hilbert ring.

(2) The hypothesis of the Theorem that the smooth points are dense in max-spec R cannot simply be dropped: there is a 2-dimensional Noetherian regular factorial ring R whose maximal ideals form a countable set, say $\{m_1, m_2, \ldots\}$, such that $\bigcap_i m_i$ is a nonzero, principal ideal (f), whose generator f is in the ith power of m_i for all i. Setting $\bar{R} = R/(f^2)$, we see that there is no integer k such that the intersection of the kth powers of the maximal ideals of $\bar{R}$ is 0.

The example may be constructed as follows: Let $\{X_n\}$, $\{Y_n\}$ be countable families of indeterminates over an algebraically closed field K. Set:

$$f_n = X_n{}^n - Y_n^{n+1};$$

$$I_n = (f_2 - f_1, \ldots, f_n - f_1)\, K[X_1, Y_1, \ldots, X_n, Y_n];$$

$$S_n = K[X_1, Y_1, \ldots, X_n, Y_n]/I_n;$$

$$U_n = S_n - \bigcup_{i=1}^{n} (X_i, Y_i)\, S_n .$$

Then U_n is a multiplicatively closed set in S_n, and we set $R_n = U_n^{-1}S_n$. There is an obvious injection $R_n \to R_{n+1}$ which is faithfully flat. We set

$$R = \varinjlim R_n ,$$

and let f be the image of f_i in R.

One can verify that the maximal ideals of R are precisely the ideals (X_i, Y_i), and that R and f have the properties above (To prove that R is Noetherian, use Cohen's Theorem [3], noting that primes of R are either maximal, and of the form (X_i, Y_i), or of height 1, and thus principal). The ideal (f) is a prime. Note that R is not pseudogeometric; the integral closure of $R/(f)$ is not a finite $R/(f)$-module.

(3) A different approach to the proof of the Theorem could be obtained by proving some kind of "Uniform Artin-Rees Theorem," which we pose as a problem:

Problem. Let R be an affine ring, and suppose that $M \subset N$ are finitely generated R-modules. Is there an integer k_0 such that for all $k > k_0$ and all maximal ideals m of R

$$M \cap m^k N = m^{k-k_0}(M \cap m^{k_0} N)\,?$$

Of course remark 2) shows that this could not be true for all rings over an algebraically closed field.

Proof of the Theorem. Let $M_i = P^i M$ for $0 \leqslant i \leqslant e$. Since $(M_i/M_{i+1})_P$ is an R_P/P_P-vectorspace, we can choose an element $f \in R - P$ such that each $(M_i/M_{i+1})_f$ is $(R/P)_f$-free.

We now claim that for any $f \in R - P$, it suffices to prove the corresponding Theorem for the ring R_f, the set $\mathcal{N}_f = \{\mathfrak{m}R_f \mid \mathfrak{m} \in \mathcal{N}, f \notin \mathfrak{m}\}$ of maximal ideals of R_f and the finitely generated R_f-module M_f. For,

$$\bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m} = \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}_f = \Big(\bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}\Big)_f = P_f ,$$

and M_f is P_f coprimary, so the hypothesis of the Theorem is satisfied, and, on the other hand $M \subset M_f$ and

$$\bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}^e M \subset \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}^e M_f ,$$

so if the latter module is 0, the former is as well.

Thus we may assume that each M_i/M_{i+1} is R/P-free from the outset. Under this hypothesis we will show

$$\mathfrak{m}^e M \cap M_i \subset \mathfrak{m} M_i \tag{***}$$

for each M_i and each $\mathfrak{m} \in \mathcal{N}$.

Once this is established, the Theorem will follow at once, since if $x \in M_i \cap \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}^e M$, then by (***), $x \in \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m} M_i$, so $x + M_{i+1} \subset \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}(M_i/M_{i+1}) = 0$, so $x \in M_{i+1} \cap \bigcap_{\mathfrak{m}\in\mathcal{N}} \mathfrak{m}^e M$, and, continuing in this way, $x = 0$.

It remains to prove (***). Because of the behavior of sets of associated primes with respect to exact sequences,

$$\operatorname{Ass}(M/\mathfrak{m}M_i) \subset \operatorname{Ass}(M/M_i) \cup \operatorname{Ass}(M_i/\mathfrak{m}M_i) = \{P, \mathfrak{m}\},$$

So it suffices to prove (***) after localizing at $\mathfrak{m}$.

We will now change notation, and write $R, M,\ldots$ for $R_{\mathfrak{m}}, M_{\mathfrak{m}}, \ldots$. Since R/P is a regular local ring, $\mathfrak{m}/P$ is generated by a regular sequence $\bar{x}_1, \ldots, \bar{x}_d$. Lifting these elements to $x_1, \ldots, x_d \in R$, we see that $x_1, \ldots, x_d$ is an M_i/M_{i+1}-regular sequence for each i. It follows at once that $x_1, \ldots, x_d$ is an M/M_i-regular sequence for each $i > 0$, and thus that

$$(x_1, \ldots, x_d)M \cap M_i = (x_1, \ldots, x_d)M_i .$$

On the other hand, $\mathfrak{m} = P + (x_1, \ldots, x_d)$, so

$$\begin{aligned}\mathfrak{m}^e M \cap M_i &= \Big(\sum_{j+k=e} P^j(x_1, \ldots, x_d)^k M\Big) \cap M_i \\ &\subset (P^e M + (x_1, \ldots, x_d)M) \cap M_i \\ &= (x_1, \ldots, x_d)M \cap M_i \\ &= (x_1, \ldots, x_d)\, M_i \\ &\subset \mathfrak{m} M_i ,\end{aligned}$$

as required for (***). This completes the proof.

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