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On the Number of Generators of Ideals in Local Cohen–Macaulay Rings

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INTRODUCTION

In this paper we are concerned with bounds for the number of generators of an ideal I, for which we write $\mu(I)$, in a local Cohen-Macaulay ring. Our model is the theorem of Lipman:

Of course, for 2-dimensional rings sufficiently high powers of the maximal ideal require arbitrarily large numbers of generators, so there can be no universal bound for the number of generators of an ideal. However our main results (Theorems 1 and 3) give bounds for ideals of fixed colength, and ideals with a fixed number of irreducible components primary to the maximal ideal in a 2-dimensional Cohen-Macaulay ring. We also show how to extend these results to results on the number of generators of ideals primary to the maximal ideal in higher dimensional Cohen-Macaulay rings.

For a general survey of this and related problems, we refer the reader to the notes [Sally 2].

Related estimates connecting multiplicity and number of generators have been obtained previously by J. Becker; see [12] and, particularly, [13].

THEOREM. *Let R be a 1-dimensional local Cohen-Macaulay ring of multiplicity e.*

If I is an ideal of R, then $\mu(I) \leqslant e$. In general the multiplicity of an ideal I, primary to the maximal ideal of R, on an R-module M, will be written as $e(I; M)$; it is $(\dim R)!$ times the coefficient of degree d of the polynomial function $H_{I,M}(\nu) = \text{length}\, I^{\nu}M/I^{\nu+1}M$. The multiplicity of R is the multiplicity of the maximal ideal on R.

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1. Generating Ideals in a 2-Dimensional Cohen-Macaulay Ring

Our main theorem is:

Theorem 1. *Let R be a 2-dimensional Cohen-Macaulay ring with maximal ideal $\mathcal{M}$, and let $e = e(\mathcal{M}; R)$ be the multiplicity of R. If I is an ideal of R with irreducible decomposition $I = \cap I_\alpha$ where t of the I_α are $\mathcal{M}$-primary, then*

$$\mu(I) \leqslant e + et.$$

This, and some other results will be deduced from the following Lemma from [Rees 2] in which we write $\lambda(M)$ for the (possibly infinite) length of an R-module M.

Lemma 2. *With hypotheses as in Theorem 1, assume in addition that R has an infinite residue class field, and let $x_1, x_2 \in R$ be a system of parameters with $e(x_1, x_2; R) = e(\mathcal{M}; R)$. (Such systems exist by [Northcott-Rees].) Then $\mu(I) \leqslant e(R) + \lambda(I\colon (x_1, x_2)/I)$.*

Proof of Lemma 2. For any R-module M, set $H_i(M) = \operatorname{Tor}_i^R(R/(x_1x_2), M)$. Then $\lambda H_i(M) < \infty$, and $H_i(M) = 0$ for $i > 2$. By [Serre], $e(M) = e(x_1, x_2; M) = \sum_0^2 (-1)^i \lambda H_i(M)$.

Now $H_0(M) = M/(x_1, x_2)M$, so $\lambda H_0(M) \geqslant \mu(M)$.

Further, if $I \subset R$, then the long exact sequence in Tor associated to $0 \to I \to R \to R/I \to 0$ yields $H_2(I) = H_3(R/I) = 0$, and $H_1(I) = H_2(R/I)$. Using the Koszul complex to calculate $\operatorname{Tor}_A{}^R(R/(x_1, x_2), -)$, we see that $H_2(R/I) = \operatorname{ann}_{R/I}((x_1, x_2) + I/I) = (I\colon (x_1, x_2))/I$. Putting this together we get

$$\mu(I) \leqslant \lambda H_0(I) = e(x_1, x_2, I) + \lambda(I\colon (x_1, x_2)/I).$$

It now suffices to note that since $e(\ )$ is additive on short exact sequences, $e(x_1, x_2, I) \leqslant e(R)$ (in fact, equality holds if I contains a non-zerodivizor).

Proof of Theorem 1. We may harmlessly assume that the residue field of R is infinite, for example by passing to a polynomial ring $R[x]$ localized at $\mathcal{M}R[x]$; under this faithfully flat extension the number of generators, multiplicities, etc., are preserved. Using the notation of Lemma 1, it thus suffices to show

$$et \leqslant \lambda[(I\colon (x_1, x_2))/I].$$

However, $(I\colon (x_1, x_2))/I = \operatorname{Hom}(R/(x_1, x_2), R/I)$, and $t = \lambda \operatorname{Hom}(R/\mathcal{M}, R(I))$. By induction on length it is easy to see that if M is an R-module, then

$$\lambda \operatorname{Hom}(M, R/I) \leqslant \lambda(M)\, \lambda(\operatorname{Hom}(R/\mathcal{M}, R(I)).$$

Since $\lambda(R/(x_1, x_2)) = e$, the desired formula results if we take $M = R/(x_1, x_2)$.

Remark. Using the ideals constructed in [Buchsbaum-Eisenbud], one can exhibit a 2-dimensional local ring of multiplicity $(n - 1)/2$ and a primary ideal with $t = 1$ requiring $n - 1$ generators for every odd n. Thus Theorem 1 is, in a certain sense, best possible. See [Sally 2] for details.

In the same context, Lemma 2 may be exploited in other ways. For example, if I is primary, then since

$$\lambda(I: (x_1, x_2)/I) \leqslant \lambda(R/I),$$

we get $\mu(I) \leqslant 2(R) + \lambda(R/I)$. (In this case the t of Theorem 1 is $\lambda(\text{socle } R/I)$, so we get an incomparable result.) This idea can be used more precisely to obtain

THEOREM 3. *Let R be a 2-dimensional Cohen-Macaulay ring, and let I be an ideal which is primary to the maximal ideal. Then*

$$\mu(I) \leqslant \tfrac{3}{2}e(R) + \tfrac{1}{2}\lambda(R/I).$$

Proof of Theorem 3. As with the proof of Theorem 1, we may assume that the residue class field of R is infinite and apply Lemma 2. Note that

$$\lambda(I: (x_1, x_2)/I) = \lambda(R/I) - \lambda(R/I: (x_1, x_2)). \tag{1}$$

But $(x_1, x_2) + I/I$ has two generators and is annihilated by I: (x_1, x_2), so

$$\lambda((x_1, x_2) + I/I) \leqslant 2\lambda R/(I: (x_1, x_2)),$$

whence, using (1)

$$\begin{aligned} \lambda(I: (x_1, x_2)/I) &\leqslant \lambda(R/I) - \tfrac{1}{2}\lambda((x_1, x_2) + I/I) \\ &= \lambda(R/I) - \tfrac{1}{2}\lambda(R/I) + \tfrac{1}{2}\lambda(R/(x_1, x_2) + I) \\ &\leqslant \tfrac{1}{2}(\lambda(R/I) + \lambda R/(x_1, x_2) + I) \\ &\leqslant \tfrac{1}{2}(\lambda(R/I) + e(R)). \end{aligned}$$

With Lemma 2, this completes the proof.

Remark. Using information about the kernel of the map $R/I: (x_1, x_2)^2 \to (x_1, x_2) + I/I$, one can obtain the estimate

$$\mu(I) \leqslant \tfrac{3}{2}e(R) + \tfrac{1}{2}\lambda(R/I) - \tfrac{1}{2}\lambda R/I: (x_1, x_2)^2.$$

But we have not been able to find an attractive estimate for this last term.

2. HIGHER DIMENSIONAL EXTENSIONS

There are two obvious methods for extending these results to higher dimensional rings. If one treats ideals I in a local Cohen-Macaulay ring R such that depth $R/I \geqslant \dim R - 2$, then factoring out a sufficiently general regular sequence from R/I will not affect $\mu(I)$ or $e(R)$, and one can use Theorem 1. More interesting, perhaps, is the case in which I is primary to the maximal ideal in a d-dimensional Cohen-Macaulay ring. The standard technique is to factor out $d - 1$ or $d - 2$ "sufficiently general" parameters contained in I, and then to use Theorems 1 or 3 or the earlier result that in a 1-dimensional Cohen-Macaulay ring, the number of generators of an ideal is bounded by the multiplicity of the ring.

With this method, the whole interest centers around the question of the minimum multiplicity obtainable for a ring of the form $R/(f_1,\ldots,f_k)$, when $f_1,\ldots,f_k \in I$, $k = d - 1$ or $d - 2$. One estimate is to use the "degree of nilpotency of R/I; if the nth power of the maximal ideal $\mathscr{M}$ of R is contained in I, then taking the f_i to be the nth powers of a system of general parameters for R, we obtain $e(R/(f_1 \cdots f_k)) = n^k \cdot e(R)$. In particular, $n \leqslant \lambda(R/I)$, so we obtain $e(R/(f_1,\ldots,f_k)) \leqslant \lambda(R/I)^k \cdot e(R)$ [Sally 1]. A different approach, expressing the minimum multiplicity in terms of the certain valuations, may be found in [Rees 1 and 3]. Here we adopt a third course, using the following result of [Teissier]:

LEMMA 4. *Suppose R is a local Cohen-Macaulay ring containing an infinite field and that I is an ideal of R which is $\mathscr{M}$-primary. Then for any $k \leqslant \dim R = d$, there exists a regular sequence $f_1,\ldots,f_k \in I$ with*

$$e(R/(f_1 \cdots f_k)) \leqslant e(I; R)^{k/d}\, e(R)^{d-k/d}.$$

Putting these ideas together, we get

THEOREM 5. *Let R be a local d-dimensional Cohen-Macaulay ring, and let I be an ideal which is primary to the maximal ideal. Then*

(a) $\mu(I) \leqslant e(I; R)^{(d-1)/d}\, e(R)^{1/d} + d - 1$ *for* $d \geqslant 1$.

(b) $\mu(I) \leqslant e(I; R)^{(d-2)/d}\, e(R)^{2/d} + \lambda(R/I) + d - 2$ *and*
$\mu(I) \leqslant \frac{3}{2}e(I; R)^{(d-2)/d}\, e(R)^{2/d} + \frac{1}{2}\lambda(R/I) + d - 2$ *for* $d \geqslant 2$.

(c) *If the length of the socle of R/I is t, then*
$\mu(I) \leqslant e(I; R)^{(d-2)/d}\, e(R)^{2/d} \cdot (t + 1) + d - 2$ *for* $d \geqslant 2$.

Proof. All these results follow, by Lemma 4 and the argument preceding it, from the corresponding results in the case $d = 1$ (for a; this is the theorem of Rees given in the introduction) or $d = 2$ (for b, use Theorem 3 and the argument before it, for c use Theorem 1). One drawback of Theorem 5 is that the number $e(I; R)$ is almost as mysterious as the number $\mu(I)$. This disadvantage can be mitigated by using the "degree of nilpotency" of R/I as above, or by applying the following inequality of [Lech]:

LEMMA 6. *If R is a local ring R of dimension d, and I is an ideal which is primary to the maximal ideal, then*

$$e(I; R) \leqslant d!\, e(R)\, \lambda(R/I).$$

Combining this with Theorem 5, we can obtain results relating $\mu(I)$ ro d, $e(R)$, $\lambda(R/I)$ and t. For example, from (a), we get

$$\mu(I) \leqslant (d!\, \lambda(R/I))^{(d-1)/d}\, e(R) + d - 1,$$

for all $d \geqslant 1$. It is interesting to compare this to the result of [Sally 1], $\mu(I) \leqslant \lambda(R/I)^{d-1} \cdot e(R) + d - 1$, obtained in the same way, but without Lemmas 4 and 6.

REFERENCES

1. D. A. BUCHSBAUM AND D. EISENBUD, Algebra Structures for resolutions and some structure theorems for ideals of codimension 3, *Amer. J. Math.* **99** (1977), 447–485.
2. C. LECH, Note on multiplicities of ideals, *Ark. Mat.* **4**, No. 8 (1960/1963).
3. J. LIPMAN, Stable ideals and arf rings, *Amer. J. Math.* **93** (1971), 649–685.
4. D. G. NORTHCOTT AND D. REES, Reductions of ideals in local rings, *Proc. Cambridge Philos. Soc.* **50** (1954), 145–158.
5. D. REES, Degree functions in local rings, *Proc. Cambridge Philos. Soc.* **57** (1961), 1–7.
6. D. REES, Estimates for the minimum number of generators of modules over Cohen–Macaulay rings, unpublished.
7. D. REES, Degree functions in two-dimensional normal local domains, *Symposia Math.* **8** (1972).
8. J. SALLY, Bounds for the number of genrators of Cohen–Macaulay ideals, *Pacific J. Math.*, in press.
9. J. SALLY, Numbers of generators of ideals in local rings, to appear.
10. J.-P. SERRE, "Algèbre Locale-Multiplicités," Lecture Notes in Mathematics, No. 11, Springer–Verlag, New York, 1975.
11. B. TEISSIER, Sur une inégalité à la Minkowski pour les multiplicités. (Appendix to "An algebraic formula for the degree of a C^∞ map germ," by. D. Eisenbud and H. Levine, *Ann. of Math.* (1977).
12. J. BECKER, Parametrizations of analytic varieties, *Trans. Amer. Math. Soc.* **183** (1973), 265–292.
13. J. BECKER, On the boundedness and unboundedness of the number of generators of ideals and multiplicity, *J. Algebra* **48** (1977), 447–453.