Ideals with a regular sequence as syzygy

DAVID EISENBUD and CRAIG HUNEKE

We sketch an alternate approach to Proposition 2, reducing it to results of Huneke and Ulrich [H-U] and Kustin [Ku] (results similar to those of Kustin were also obtained by M. Stillman). In [H-U] the authors work over a ring containing a field, but the results are general, and are done explicitly without this hypothesis in [Ku].

Assume that $R$ is a local Noetherian ring, that $x_1, \ldots, x_n$ is a regular sequence in $R$ and that $f_1, \ldots, f_n$ are elements of $R$ satisfying the relation

\[(*) \quad x_1 f_1 + \ldots + x_n f_n = 0.\]

We further set $I = (f_1, \ldots, f_n)$ and suppose that the grade of $I$ is $n - 1$, the largest possible value.

If $f$ is a form in $k[x_1, \ldots, x_n]$ defining a nonsingular hypersurface, and if $\text{char}(k)$ divides the degree of $f$, then Euler’s relation shows that these hypotheses are satisfied by the partial derivatives of $f$ in the localization of $k[x_1, \ldots, x_n]$.

**Theorem.** If $\text{grade}(I) = n - 1$, then

(i) if $n$ is odd, $R/I$ is perfect of Cohen-Macaulay type 2.

(ii) if $n$ is even, there exists an element $f \not\in I$ such that

$\quad I : (x_1, \ldots, x_n) = (I, f),$

and $R/(I, f)$ is perfect of Cohen-Macaulay type 1.

**Proof:** The most interesting point is the identity of the element $f$ : the relation $(*)$ shows that the vector $(f_i)$ is a linear combination of the syzygies of the $x_i$. Since the $x_i$ form a regular sequence, their syzygies are given by the first map of the Koszul complex $k : \Lambda^2 R^n \to \Lambda^1 R^n$, so there exists a skew-symmetric matrix $A$ such that $(f_i) = A(x_j)$. The element $f$ is then the Pfaffian of $A$.

The result follows by specialization from the generic case, which is treated in [H-U], 5.8, 5.9 and 5.12, and in [Ku].

**Corollary.** If $R$ is regular, $x_1, \ldots, x_n$ generate the maximal ideal, and $g$ is an element of $R$ such that $\text{ht}(I, g) = n$, then the socle of $R/(I, g)$ is two-dimensional.
Proof: If $n$ is odd, the corollary follows at once from (i). If $n$ is even, it follows from (ii) because $g$ must be a nonzero divisor mod$(I,f)$.

Graded free resolutions for the generic forms of the ideals $I$ and $(I,f)$ as in the Theorem can be found in [Ku], Theorem 6.3. By local duality, this gives the degrees of the sode elements in the corollary (alternatively, one can use linkage, as was done in [H-U]). Applying this to the case of partial derivatives of the equation of a nonsingular hypersurface, one recovers the degree results of Beauville.

BIBLIOGRAPHIE


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Arnaud Beauville
Université Paris-Sud
Orsay