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Nets of Alternating Matrices and the Linear Syzygy Conjectures*

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In this paper we classify the 3-dimensional vector spaces (nets) of 4×4 and of 5×5 alternating matrices over an algebraically closed field. We apply the second classification to check our "linear syzygy conjectures" for modules over the polynomial ring of dimension 5. © 1994 Academic Press, Inc.

Contents.

INTRODUCTION

The classification of 2-dimensional spaces of matrices over an algebraically closed field, that is, the theory of "matrix pencils" of Weierstrass [20] and Kronecker [17], is an extremely useful chapter in linear algebra; one modern form is the classification of coherent sheaves on the projective line, and it is also used in the study of differential equations, both from a theoretical point of view and in numerical studies (see, for example,

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Gantmacher [7, Chap. 12] for an exposition; and Kågström and Ruhe [15] for a survey of some more recent activity.) The fact that it can be carried over easily to a classification up to similarity of a 2-dimensional spaces of symmetric or alternating (that is, skew-symmetric with zero diagonal) matrices is also well known (Gantmacher [7, Thm. 12.6].) In some cases the classification is finite; in other cases it resembles the classification of single matrices up to similarity by Jordan Normal form.

It is rather natural to ask whether a classification is possible for any cases of higher dimensional spaces of matrices, and indeed to ask for the classification itself. An arithmetic comparison of the dimensions of the varieties of spaces of matrices with the dimensions of the groups whose orbits are being classified shows that there can only be a few such cases. More generally, Kac [14, Table II] has listed all the connected linear groups acting irreducibly in such a way that every level set for the invariants contains only finitely many orbits. If we restrict ourselves to linear spaces of matrices with the natural group actions, then the only cases beyond the classical ones (single matrices and pencils) where there are only finitely many orbits are:

(1) 3-dimensional spaces of alternating 4×4 matrices;

(2) 3-dimensional spaces of alternating 5×5 matrices;

(3) 4-dimensional spaces of alternating 6×6 matrices.

(Note that the case corresponding to 1) appears in Kac's table as $SL_3 \otimes SO_6$ instead of $SL_3 \otimes \Lambda^2 SL_4$.)

In the more general case, where there are finitely many orbits in each level set of the invariants, one also has 3-dimensional families of 3×3 symmetric matrices and 6×6 alternating matrices, and 5-dimensional families of alternating 5×5 matrices (for an older account of some cases of skew-symmetric trilinear forms see the last chapter of Gurevich [12] and the references therein).

Sections 1 and 2 of this paper are devoted to carrying out the classification in Cases (1) and (2). In Case (1) it turns out that there are just 5 classes, and the classification is rather easy.

In Case (2), we say that a space of matrices is *degenerate* if (with respect to a suitable basis) all its elements have a common row and column of zeros. The degenerate spaces of matrices in (2) are classified by (1). Leaving aside the degenerate spaces, there are 12 classes if the characteristic of the base field is not 2, and a thirteenth in characteristic 2. This classification is the main result of this paper.

We were lead to study these classifications in order to check a special case of the "linear syzygy conjectures" introduced in our previous paper. These concern the question of which graded modules over a polynomial ring $S = F[x_1, ..., x_v]$, where F is a field, have long chains of linear syzygies. More precisely, we say that a graded S-module M, generated in degree 0, has a *linear* kth *syzygy* if one of the generators of the kth module G_k in its minimal graded free resolution

$$\rightarrow \cdots \rightarrow G_k \rightarrow \cdots \rightarrow G_1 \rightarrow G_0 \rightarrow M \rightarrow 0$$

has degree k (the smallest possible number.) The conjectures say, roughly, that a module with a kth linear syzygy must have many elements killed by linear forms if k is not too small compared to the number of generators of M. See our paper for a detailed exposition. Related ideas have been studied by many people (for example, Ballico and Geramita [1], Eisenbud and Goto [3], Ellia and Hirshowitz [4], Green [8], Green and Lazarsfeld [9, 10], Herzog, Vasconcelos, and Simis [13], Koh and Stillman [16], Schreyer and Kempf [18, 19]) most recently for their significance in algebraic geometry.

The connection of linear syzygies with the classification of linear spaces of matrices is that an m-dimensional subspace of the kth exterior power of a v-dimensional vectors space V corresponds, in a simple way explained in our previous paper, to a "minimal" example of an S-module with m generators having a linear kth syzygy. If $k = 2$ or $k = v - 2$, so that $\Lambda^k V = \Lambda^2 V^*$, such a subspace corresponds to an m-dimensional space of skew symmetric matrices. We showed in our previous paper that it would be enough to check the conjectures for these minimal examples—indeed, for the minimal examples with $m = k$. We also verified all our conjectures in various special cases, including all the cases for $v \leqslant 5$ except the one with $m = 3$, $k = 3$. Since $v - k = 2$ in this case, the minimal examples here correspond to the spaces of matrices targeted in the classification problem (2) above.

Our solution of the classification problem (2) above thus allows us to check our conjectures over any finite field, and we get:

THEOREM. *The linear syzygy conjectures hold for all modules over a polynomial ring in 5 or fewer variables, at least over algebraically closed fields of some characteristics.*

Proof Sketch. We may apply the classification of spaces of matrices (2). The 12 or 13 types produced correspond 3-generator modules over $k[x_1, ..., x_5]$, and it is enough to verify that each of these modules satisfies the conjectures. This can be done by computation over any field of the same characteristic, though the computations involved would be quite difficult to undertake by hand. Fortunately they are well suited to the program Macaulay of Bayer and Stillman [2], and we have made an extensive investigation of them using that program. Some of the results are reported in Section 3. In particular, all the linear syzygy conjectures are confirmed, proving the theorem. The restriction to "some characteristics," which is presumably unnecessary, is there because the computations were done over particular fields, including the fields with 31991 elements and with 2 elements. ∎

Since the linear syzygy conjecture concerns a lower bound on the dimension of a certain variety defined from a given module, the result should be the same for all but finitely many characteristics at worst; it would be very bad luck indeed if 31991 and the other primes we have checked were all among these bad characteristics.

In Section 3 we explain some of the "experimental" information that we have gathered about the 13 modules produced from the 13 cases of the classification in Section 2, and in Section 4 we exhibit parts of the Macaulay program we used.

Part of the classification of spaces as in (2) was worked out in conversations with Joe Harris; we are grateful to him (as usual) for his help. We are also indebted to Mike Stillman. Our whole linear syzygy project grew from discussions with him. In addition, our conjectures were based from the first on evidence produced by his and Dave Bayer's program Macaulay [2], and we would not even have begun the work in this paper had we not felt that Macaulay would bridge the gap from the classification to the verification of the conjectures.

1. Classification of Nets of Trilinear Alternating Forms in 4 Variables

Let V be a vector space of dimension 4 over an algebraically closed field F. In this section we will classify 3-dimensional subspaces N of $\Lambda^2 V$ (these correspond to spaces of alternating maps $V^* \to V$, or, again, to alternating matrices whose entries are linear forms in the polynomial ring $F[N^*]$) up to the action of $GL(N) \times GL(V)$. It turns out that there are 5 distinct examples.

We denote by $\mathbb{P}(\Lambda^2 V)$ the projective space of lines in $\Lambda^2 V$ and by $\mathbb{P}(N) \cong \mathbb{P}^2$ the linear subspace of $\mathbb{P}(\Lambda^2 V)$ associated to N. Of course it is an equivalent problem to classify the 2-dimensional projective subspaces $\mathbb{P}(N)$ in $\mathbb{P}(\Lambda^2 V)$. The elements of N may be regarded as skew symmetric maps from V^* to V. Of course the rank of a map is the same as the rank of a scalar multiple of the map, so we may speak of the rank of an element of $\mathbb{P}(N)$. We write $\mathbb{P}(N)_2$ for the subscheme of elements of rank 2, which may also be regarded as the intersection of $\mathbb{P}(N)$ and the Grassmannian of lines in $\mathbb{P}(V)$, in its natural embedding in $\mathbb{P}(\Lambda^2 V)$. The equation of $\mathbb{P}(N)_2$

is the pfaffian of the alternating matrix of linear forms over $F[N^*]$ representing N. This equation is a quadratic form on $\mathbb{P}(N)$. We thus have the following possibilities for $\mathbb{P}(N)_2$:

Case I: $\mathbb{P}(N)_2$ is a nonsingular conic.

Case II: $\mathbb{P}(N)_2$ is a union of two distinct lines.

Case III: $\mathbb{P}(N)_2$ is a double line.

Case IV: $\mathbb{P}(N)_2 = \mathbb{P}(N)$ and the transformations in N have no common kernel.

Case V: $\mathbb{P}(N)_2 = \mathbb{P}(N)$ and the transformations in N have a common kernel.

THEOREM 1.1. *These 5 cases each correspond to unique orbits of 3-dimensional subspaces N in $\Lambda^2 V$.*

We will exploit the fact that in each of the first three cases the scheme $\mathbb{P}(N)_2$ spans—and thus determines—the space N, so that the classification really is the classification of the conics on the Grassmannian of lines in $\mathbb{P}^3$. For Case I, a nonsingular conic on the Grassmannian is either the set of lines in one ruling of a nonsingular quadric surface in $\mathbb{P}^3$ or the lines on a singular quadric surface; the second of these possibilities does not occur in Case I because the plane spanned by this sort of conic lies entirely in the Grassmannian (it is the plane of lines through the vertex) and is thus absorbed in Case IV. Any line on the Grassmannian corresponds to the lines contained in a fixed plane in $\mathbb{P}^3$ and passing through a fixed point in that plane; the various configurations of these lead to Cases II and III.

Cases IV and V correspond, of course, to the classification of 2-planes in the Grassmannian of lines in $\mathbb{P}(V)$. Quite generally, the classification of (maximal) linear subspaces of a Grassmannian is well-known, and is described in the following Lemma. We do not know a reference, so we sketch the proof (see Griffiths and Harris [11, p. 787] for the special case of the Grassmannian of lines). We will apply the Lemma again in the next section.

LEMMA 1.2. *Let V be a finite dimensional vector space, and let*

$$G = \mathrm{Gr}(k, \mathbb{P}(V)) \subset \mathbb{P}(\Lambda^{k+1} V) = \mathbb{P}^r.$$

(i) *Let Λ denote the line of $\mathbb{P}(\Lambda^{k+1} V)$ through distinct points α_1, $\alpha_2 \in G$ and write U, $U' \in \mathbb{P}(V)$ for the intersection and span of the k-spaces corresponding to α_1 and α_2. We have*

$$\Lambda \in G \quad \textit{iff} \quad \dim U = k-1 \ \textit{iff} \ \dim U' = k+1,$$

in which case $\Lambda = \{\alpha \mid U \subset \alpha \subset U'\}$. *With respect to a suitable basis* e_i *of* V, *the space* Λ *thus consists of the linear combinations of the elements*

$$e_1 \wedge \cdots \wedge e_{k-1} \wedge e_k \quad \text{and} \quad e_1 \wedge \cdots \wedge e_{k-1} \wedge e_{k+1}.$$

(ii) *Each maximal linear subspaces of* $\mathbb{P}^r$ *contained in G is of the form*

$$\{\alpha \mid W \subset \alpha \text{ for some } (k-1)\text{-space } W\}$$

or

$$\{\alpha \mid \alpha \subset W \text{ for some } (k+1)\text{-space } W\}.$$

Proof. To deduce (ii) from (i) we use induction on the dimension of L. Since (i) is the case $\dim L = 1$, we may assume that $\dim L > 1$. Let Λ be a line in L and N a codimension 1 subspace such that $L = \Lambda + N$. By induction hypothesis N satisfies condition (a) or (b). Since two conditions correspond to each other in the isomorphism $\mathbb{P}(\Lambda^{k+1} V) \cong \mathbb{P}(\Lambda^{v-k-1} V^*)$, we may assume that N satisfies the condition (a). Let W be a $(k-1)$-plane such that $N \subset \{\alpha \mid W \subset \alpha\}$ and let U be a $(k-1)$-plane such that $\Lambda \subset \{\beta \mid U \subset \beta\}$. Since $L \cap N \neq \emptyset$, $\dim U \cap W \geqslant k-2$. If $\dim U \cap W = k-1$, then $U = W$ and L satisfies condition (a). Now assume that $\dim U \cap W = k-2$. If $\alpha \in N$ and $\beta \in \Lambda$ are distinct, then $\dim \alpha \cap \beta = k-1$ by *i*). Since this is only possible when Λ and N are contained in $\{\gamma \mid \gamma \subset V'\}$ for some $(k+1)$-plane V', L satisfies the condition (b) and we are done.

We now prove (i). One direction is easy: It is clear that $\dim U = k-1$ iff $\dim U' = k+1$. If these conditions hold we can take a spanning set for α_i of the form $e_1, \ldots, e_k, f_i$, where $e_1, \ldots, e_k$ span U, and the line spanned by the corresponding points in $\mathbb{P}(\Lambda^{k+1} V)$ is

$$\{e_1 \wedge \cdots \wedge e_k \wedge (sf_1 + tf_2) \mid (s, t) \in \mathbb{P}^1\}$$

which obviously lies in the Grassmannian, as the set

$$\{\alpha \in \mathrm{Gr}(k, \mathbb{P}(V)) \mid U \subset \alpha \subset U' = \langle U, f_1, f_2 \rangle\}.$$

For the converse, suppose that Λ is contained in the Grassmannian. Choose $\beta_1, \beta_2 \in \Lambda$ such that $U = \beta_1 \cap \beta_2$ is of the largest possible dimension. If that dimension is $k-1$ we are done by the construction above, so we may assume that it is $< k-1$. Let α be a point of Λ distinct from β_1 and β_2. We will derive a contradiction by showing that there is a hyperplane H in $\mathbb{P}(\Lambda^{k+1} V)$ containing β_1 and β_2 but not α.

Recall that a hyperplane section of the Grassmannian consists of all those k-planes meeting some fixed plane γ of codimension $k+1$ in $\mathbb{P}(V)$. Thus we need only find a γ meeting β_1 and β_2 but not α. If $\alpha \cap \beta_1 \cap \beta_2 \neq U$, then a general plane of codimension $k+1$ through a point of $U-\alpha\cap\beta_1\cap\beta_1$ will obviously satisfy the condition, so we may assume that α contains U as well. Thus it is enough to solve the problem mod U, and we will change notation and assume for simplicity that U was empty from the outset. Under these assumptions we must derive a contradiction if $k>0$.

Let $\pi\colon \mathbb{P}(V)\to\mathbb{P}(V/\alpha_1)$ be the projection map. Because of our assumptions, π is well-defined and one-to-one on each of α and β_2. If there is a point of $\pi(\beta_2)$ which is not contained in $\pi(\alpha)$ then a general plane of codimension $k+1$ through that point in $\mathbb{P}(V/\beta_1)$ will lift back to a plane of codimension $k+1$ in $\mathbb{P}(V)$ with the desired property, so we may assume that $\pi(\alpha)=\pi(\beta_2)$. Let γ' be a general plane of codimension k in $\mathbb{P}(V/\beta_1)$, so that γ' meets $\pi(\alpha)$ and $\pi(\beta_2)$ in a single point. The preimage $\pi^{-1}(\gamma')$ is a codimension k plane of $\mathbb{P}(V)$ containing β_1 and meeting each of α and β_2 in a single (distinct) points, because π is one-to-one on each of α and β_2 and $\alpha\cap\beta_2=\varnothing$. Thus there is a codimension 1 plane γ in $\pi^{-1}(\gamma')$ which contains a point of β_2 but no pointy of α. This plane γ meets β_1 in a plane of codimension $\leqslant 1$ in β_1—and thus in a nonempty set, as required, as long as $k>0$. ∎

We now give an algebraic proof of the Theorem:

Cases I *and* II. Let $\alpha=e_1\wedge e_2$ and $\beta=e_3\wedge e_4$ be two distinct elements of $\mathbb{P}(N)_2$ such that the line joining α and β is not contained in $\mathbb{P}(N)_2$. Then $\{e_1,e_2,e_3,e_4\}$ is linearly independent by Lemma 1.2(i). Let $\gamma=\sum_{1\leqslant i<j\leqslant 4} c_{ij}e_i\wedge e_j$ be a third vector such that α, β, γ forms a basis of N. Subtracting multiples of α and β, we may assume that $c_{12}=0$ and $c_{34}=0$. We may represent N as

$$\begin{pmatrix} xA & zB \\ -zB^{t} & xA \end{pmatrix},$$

where

$$A=\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and B is a nonzero 2×2 scalar matrix. Since we are free to use any row and column operations on zB, we have two cases depending on the rank of B.

Case I, rank $B=2$:

$$\begin{pmatrix} 0 & x & z & 0 \\ -x & 0 & 0 & z \\ -z & 0 & 0 & y \\ 0 & -z & -y & 0 \end{pmatrix}.$$

The pfaffian is $xy-z^2$ and we have a smooth conic.

Case II, rank $B = 1$:

$$\begin{pmatrix} 0 & x & z & 0 \\ -x & 0 & 0 & 0 \\ -z & 0 & 0 & y \\ 0 & 0 & -y & 0 \end{pmatrix}.$$

The pfaffian is xy and we have a union of two distinct lines.

Case III. Let α and β be two distinct points of the double lines $\mathbb{P}(N)_2$. By Lemma 1.2(i), we may assume that $\alpha = e_1 \wedge e_2$ and $\beta = e_1 \wedge e_3$. Let $\gamma = \sum_{1 \leqslant i < j \leqslant 4} c_{ij} e_i \wedge e_j$ be third vector such that α, β, γ forms a basis of N. Subtracting multiples of α and β, we may assume that $c_{12} = 0$ and $c_{23} = 0$. We may represent N as

$$\begin{pmatrix} 0 & x & y & a \\ -x & 0 & b & c \\ -y & -b & 0 & d \\ -a & -c & -d & 0 \end{pmatrix},$$

where a, b, c, and d are scalar multiples of z. Since the pfaffian $xd - yc + ab$ is supposed to be a square of a linear form, we must have $c = d = 0$, $a \neq 0$, and $b \neq 0$.

Case III:

$$\begin{pmatrix} 0 & x & y & z \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ -z & 0 & 0 & 0 \end{pmatrix}.$$

The pfaffian is z^2.

Cases IV *and* V. In this case we know from Lemma 1.2(ii) that N is either $e_1 \wedge V'$ or $\Lambda^2 V'$ for some 3-dimensional subspace of V. In the first of these cases we may assume that N is spanned by $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_1 \wedge e_4$:

Case IV:

$$\begin{pmatrix} 0 & x & y & z \\ -x & 0 & 0 & 0 \\ -y & 0 & 0 & 0 \\ -z & 0 & 0 & 0 \end{pmatrix}.$$

In the second of these cases, $N = \Lambda^2 V'$, we may clearly write N as Case V:

$$\begin{pmatrix} 0 & x & y & 0 \\ -x & 0 & z & 0 \\ -y & -z & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

2. Classification of Nets of Alternating Forms in 5 Variables

Let V be a vector space of dimension 5 over an algebraically closed field. In this section we will classify 3-dimensional subspaces N of $\Lambda^2 V$. Here we consider only nondegenerate subspaces (since the degenerate ones essentially fall into the classification of Section 1) and classify these up to linear automorphisms of V. It turns out that there are 12 distinct examples if the characteristic is $\neq 2$, 13 if it is $= 2$.

We use the same notation as Section 1. The equations of $\mathbb{P}(N)_2$ are the pfaffians of the 4×4 principal submatrices of the matrix of linear forms representing N. These equations are 5 quadratic forms on $\mathbb{P}(N)$. Thus regarding $\mathbb{P}(N)$ as a projective plane, $\mathbb{P}(N)_2$ is either empty, 0-dimensional, a (possibly double) line, or a conic spanning the plane (2 distinct lines or a smooth conic).

We will break the classification up by the geometry of $\mathbb{P}(N)_2$ as follows:

Case I: $\mathbb{P}(N)_2 = \varnothing$ (generic case) (1 example).

Case II: $(\mathbb{P}(N)_2)_{\text{red}}$ is 0-dimensional, and does not span the plane.

IIa: One point (4 examples in characteristic $\neq 2$, 5 in characteristic 2).

IIb: Two points (2 examples).

Case III: $(\mathbb{P}(N)_2)_{\text{red}}$ is a line (3 examples).

Case IV: $(\mathbb{P}(N)_2)_{\text{red}}$ spans the plane (2 examples).

Case I. Let A be a 5×5 alternating matrix of linear forms in 3 variables representing N. In this case the ideal of pfaffians of A has codimension 3, so A maybe obtained from the ideal of pfaffians by resolving. But the ideal of pfaffians is a 0-dimensional Gorenstein ideal, so it is the annihilator of a form of degree 2 in the divided power algebra dual to the polynomial ring in 3 variables (in char 0 we can think of this as a degree 2 differential operator.) Writing ∂_x for the dual basis element to x, etc., we may by the classification of conics write this form as $\partial_x(2) + \partial_y \partial_z$, $\partial_y \partial_z$, or $\partial_z(2)$. But in the last 2 cases, the annihilator contains the linear form x, whereas the ideal of pfaffians is generated by quadratic forms, so it is after a change of variables the annihilator of the first form. Thus there is a unique example of this type, and the matrix A is the matrix of syzygies of the ideal $(x^2-yz, xy, xz, y^2, z^2)$. Note that we are using the classification of the orbits of the action of $GL(3)$ on the quadratic forms to deduce a classification of the orbits in the space of divided powers of degree 2—this is ok because the latter space is the dual space to the former.)

We get

Case I:

$$\begin{pmatrix} 0 & 0 & 0 & x & y \\ 0 & 0 & x & y & z \\ 0 & -x & 0 & z & 0 \\ -x & -y & -z & 0 & 0 \\ -y & -z & 0 & 0 & 0 \end{pmatrix}.$$

Case II. Here $(\mathbb{P}(N)_2)_{\text{red}}$ consists of at most 2 points (else, since it does not span the plane it contains 3 collinear points, and since it is cut out by quadrics it then contains the line through them, so we are in Case III).

Case IIa. $(\mathbb{P}(N)_2)_{\text{red}}$ consists of 1 point (four examples if char $k \neq 2$, five examples in characteristic 2). The examples are distinguished by the relative position of the kernel of the unique rank 2 transformation $\gamma \in \mathbb{P}(N)$ and the conic formed by the points in $\mathbb{P}(V^*)$ corresponding to kernels of elements of a general line in $\mathbb{P}(N)$. Except for the characteristic 2 example, they are also distinguished by the scheme structure of $\mathbb{P}(N)_2$. This is the most complicated case.

For each $\alpha \in \mathbb{P}(N)-\gamma$, the kernel of α as a map from V^* to V is 1-dimensional. Define $\varphi\colon \mathbb{P}(N)-\gamma \to \mathbb{P}(V^*)$ by

$$\varphi(\alpha) = \mathbb{P}(\operatorname{Ker} \alpha).$$

If we represent N as a 5×5 alternating matrix of linear forms, then φ is given by its 4×4 pfaffians. Let Γ be a line in $\mathbb{P}(N)$ not containing γ. Since Γ is a family of maps of constant rank 4, φ is one-to-one on Γ because otherwise Γ could be represented as a 4×4 alternating matrix of linear forms and then Γ would drop rank on the zero set of the determinant.

We now regard φ as a rational map on all of $\mathbb{P}(N)$. The following gives a classification for all rational maps sharing its properties:

PROPOSITION 2.1. *Let $\varphi\colon \mathbb{P}(N) \cong \mathbb{P}^2 \to \mathbb{P}^r$ be a rational map such that* (i) *φ is defined by quadrics,* (ii) *φ has $\gamma \in \mathbb{P}(N)$ as a fundamental point but well-defined elsewhere, and* (iii) *φ is one-to-one on every line of $\mathbb{P}(N)$ not containing γ. Then φ is one of the following* 4 *maps*:

(1) *The projection p_1 of the Veronese surface from γ onto the rational normal scroll $S = S(1, 2)$ in $\mathbb{P}^4$.*

(2) *The composition p_2 of p_1 and the projection from a point of the line E_1 of self-intersection -1 on S; the image is then a quadric cone X in $\mathbb{P}^3$, with a distinguished line E_2 on it, E_2 being the image of the tangent plane to S at the projection center.*

(3) *The composition of p_2 and the projection from a point of E_2 other than the vertex; the image is $\mathbb{P}^2$.*

(4) *The composition of p_2 and the projection of X from the vertex.*

There is up to isomorphism a unique projection of each of these types.

Proof. Since φ is defined by quadrics that have a base point, φ is a projection of the Veronese surface from a point on it, which is p_1, or φ is p_1 composed with further projections. The image in S of the projection center on the Veronese is the line $E_1 \subset S$ which is the "directrix." To project further we note that the center of the projection has to be a point off S or a point on E_1 because γ is the only fundamental point of φ. Suppose first that φ were the composition of p_1 and the projection from a point off S. The image of S under projection from any point off S is a cubic surface in $\mathbb{P}^3$ with a double line; in particular, the projection is not one to one. However, the projection map is one-to-one on each ruling of S, and these rulings correspond to the lines through γ in N; since by hypothesis φ is one-to-one on each line of N not through γ, we get a contradiction. Thus if φ is not equal to p_1, it must involve a further projection from a point on E_1. The image of such a projection is a quadric cone in $\mathbb{P}^3$ and the composition of p_1 and such a projection is a map of the form p_2. If yet a third projection is necessary, it must be from a point on the line E_2 which is the image of the tangent plane to the center of the second projection since else the composition would have more fundamental points. We get Case (3) if the third projection is from a point other than the vertex and Case (4) if the projection is from the vertex. ∎

We will show that we get one example of a subspace N corresponding to each of the types of maps φ above, except in characteristic 2, where we get 2 examples corresponding to $\varphi = p_1$.

The representation of N as a matrix of linear forms depends on the choice of bases of V and N. In most cases the choice of basis of N is quite clear and in our discussion of the Case IIa, we will just describe a basis E of V. We denote by $[N]_E$ (resp. $[\alpha]_E$ for $\alpha \in \Gamma$) the matrix representation of N (resp. α) with respect to the bases E^* of V^* and E of V.

Let Γ be a line in $\mathbb{P}(N)$ not containing γ. Since Γ is a pencil of skew 2-forms of constant rank 4, we can find a basis E (see Gantmacher [7] or Eisenbud and Koh, Lemma 7.3) so that

$$[\alpha]_E = \begin{pmatrix} 0 & 0 & * & * & 0 \\ 0 & 0 & 0 & * & * \\ * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 \\ 0 & * & 0 & 0 & 0 \end{pmatrix} \qquad (*)$$

for all $\alpha \in \Gamma$.

Definitions and Notation. (1) A basis E of V is said to be normalizing Γ if $(*)$ holds.

(2) Let

$$A = \begin{pmatrix} s & t \\ u & v \end{pmatrix}$$

be a 2×2 invertible matrix. We denote by $S_2(A)$ the 3×3 matrix

$$\begin{pmatrix} s^2 & 2st & t^2 \\ su & sv + tu & tv \\ u^2 & 2uv & v^2 \end{pmatrix}.$$

(If we interpret S_2 as the degree 2 part of the symmetric algebra functor in the usual way, then this matrix should really be written as $S_2(A^*)^*$, the "second divided power of A." We will write it as $S_2(A)$ to keep the notation simple.)

(3) We will write $\langle v_1^*, ..., v_n^* \rangle$ for the subspace spanned by a set of vector $v_1^*, ..., v_n^*$ of V^*.

LEMMA 2.2. *Let A and B be invertible matrices of size 2×2 and 3×3, respectively. Then*

$$A^{-1} \begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix} B = \begin{pmatrix} L_1 & L_2 & 0 \\ 0 & L_1 & L_2 \end{pmatrix}$$

for some linear forms L_i if and only if $B = S_2(A)$.

Proof. Easy computation. Conceptually, the matrix

$$\begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix}$$

is the natural map over $F[x, y]$ corresponding to the diagonal map from $S_2(\langle x, y\rangle) \to \langle x, y\rangle \otimes \langle x, y\rangle$. The lemma follows from the fact that this does not depend on a choice of bases.... ∎

COROLLARY 2.3. *Let $E = e_i$ be a basis of V normalizing Γ. Let $D = d_i$ be a basis of V and let $P = (p_{ij})$ be the transition matrix from E to D, i.e., $e_i = \sum p_{ji} d_j$. Then D is normalizing Γ if and only if*

$$P = \begin{pmatrix} A^{-1} & C \\ 0 & S_2(A)^t \end{pmatrix},$$

where A is a 2×2 invertible matrix and B is a 2×3 matrix such that

$$A^{-1} \begin{pmatrix} x & y & 0 \\ 0 & x & y \end{pmatrix} C^t$$

is symmetric.

Proof. If E normalizes Γ, then $\langle e_1, e_2\rangle = \{\bigcap \operatorname{Im} \gamma \mid \gamma \in \Gamma\}$ and this gives the block of zeros. The rest follows from the Lemma 2.2 and by computation. ∎

Remark. The Corollary 2.3 tells us the types of base change of V which preserve the normalized form of Γ. Since we will be working with subspaces of $\mathbb{P}(V^*)$, we will use P^t, which is the transition matrix from D^* to E^*, to change basis of V^*. We would use the following two types of P^t:

(1)

$$P^t = \begin{pmatrix} (A^{-1})^t & 0 \\ 0 & S_2(A) \end{pmatrix}$$

to change bases of $\langle e_3^*, e_4^*, e_5^*\rangle$, and

(2)

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ g_1 & g_2 & 1 & 0 & 0 \\ g_2 & g_3 & 0 & 1 & 0 \\ g_3 & g_4 & 0 & 0 & 1 \end{pmatrix}$$

to change e_1^* and e_2^*.

We note that if $E = \{e_i \mid 1 \leqslant i \leqslant 5\}$ is a basis of V normalizing Γ, then $\varphi(\Gamma) \subset \mathbb{P}(\langle e_3^*, e_4^*, e_5^*\rangle) \subset \mathbb{P}(V^*)$.

LEMMA 2.4. *If Q_1 and Q_2 be two distince points of $\varphi(\Gamma) \subset \mathbb{P}(V^*)$, then there is a basis $E = \{e_i \mid 1 \leqslant i \leqslant 5\}$ of V normalizing Γ such that $Q_1 = e_3^*$ and $Q_2 = e_5^*$.*

Proof. Let $E = \{e_i \mid 1 \leqslant i \leqslant 5\}$ be a basis of V normalizing Γ and let $Q_i = s_i^2 e_3^* + s_i t_i e_3^* + t_i^2 e_3^*$, $i = 1, 2$. Then

$$A = \begin{pmatrix} s_1 & s_2 \\ t_1 & t_2 \end{pmatrix}$$

is invertible and

$$P^t = \begin{pmatrix} (A^{-1})^t & 0 \\ 0 & S_2(A) \end{pmatrix}$$

make the desired change of bases. ∎

Convention. Let E be a basis of V normalizing Γ. Let α and β be elements of Γ such that

$$[\alpha]_E = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[\beta]_E = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix}.$$

We will denote by $[N]_E$ the matrix representation of the family N with respect to the bases E of V and α, β, γ of N.

LEMMA 2.5. *If $\mathbb{P}(\operatorname{Ker} \gamma)$ contains two distinct points of $\varphi(\Gamma)$, then $\mathbb{P}(\operatorname{Ker} \gamma) \supset \varphi(\Gamma)$.*

Proof. By Lemma 2.4, we may assume that there is a basis E of V normalizing Γ such that $\langle e_3^*, e_5^* \rangle \subset \operatorname{Ker} \gamma$.

Let $\operatorname{Ker}\gamma = \langle e_3^*, e_5^*, g_1e_1^* + g_2e_2^* + g_4e_4^* \rangle$. Then

$$[\gamma]_E = \begin{pmatrix} 0 & -g_4 & 0 & g_2 & 0 \\ g_4 & 0 & 0 & g_1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ g_2 & g_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[N]_E = \begin{pmatrix} 0 & -g_4z & x & y+g_2z & 0 \\ g_4z & 0 & 0 & x-g_1 & y \\ -x & 0 & 0 & 0 & 0 \\ -y-g_2z & -x+g_1z & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 \end{pmatrix}.$$

We note that $g_1 = 0$ [resp. $g_2 = 0$] because otherwise $x - g_1z = y = 0$ [resp. $x = y - g_2z = 0$] would give a rank 2 map different from γ. Hence $\mathbb{P}(\operatorname{Ker}\gamma) = \mathbb{P}(\langle e_3^*, e_4^*, e_5^* \rangle) \supset \varphi(\Gamma)$. ∎

We now analyze the cases according to the position of $\mathbb{P}(\operatorname{Ker}\gamma) = \mathbb{P}(\langle e_3^*, e_4^*, e_5^* \rangle)$, the plane containing $\varphi(\Gamma)$.

Case IIa(i). $\mathbb{P}(\operatorname{Ker}\gamma) = \mathbb{P}(\langle e_3^*, e_4^*, e_5^* \rangle)$. In this case

$$[\gamma]_E = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and we get

Case IIa(i):

$$\begin{pmatrix} 0 & z & x & y & 0 \\ -z & 0 & 0 & x & y \\ -x & 0 & 0 & 0 & 0 \\ -y & -x & 0 & 0 & 0 \\ 0 & -y & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (x^2, xy, y^2), the ideal of the noncurvilinear triple point in the plane.

Case IIa(ii). $\mathbb{P}(\operatorname{Ker}\gamma)\cap\mathbb{P}(\langle e_3^*, e_4^*, e_5^*\rangle)=l$, a line. By Lemma 2.5, l must be tangent to $\varphi(\Gamma)$ at some point, say e_5^* using Lemma 1.3. Then $l=\mathbb{P}(\langle e_4^*, e_5^*\rangle)$ and $\operatorname{Ker} c=\langle e_4^*, e_5^*, g_1e_1^*+g_2e_2^*+g_3e_3^*\rangle$. Hence

$$[N]_E=\begin{pmatrix} 0 & g_3 & -g_2 & 0 & 0\\ -g_3 & 0 & g_1 & 0 & 0\\ g_2 & -g_1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

and

$$[N]_E=\begin{pmatrix} 0 & g_3z & x-g_2z & y & 0\\ -g_3z & 0 & g_1z & x & y\\ -x+g_2z & -g_1z & 0 & 0 & 0\\ -y & -x & 0 & 0 & 0\\ 0 & -y & 0 & 0 & 0\end{pmatrix}.$$

We note that $g_2=0$ because otherwise $x+g_2z=y=0$ would give a rank 2 map different from γ. Since $\varphi(\Gamma)\subset\mathbb{P}(\operatorname{Ker}\gamma)$ and $g_2=0$, $g_1\neq 0$ and we may assume that $g_1=1$. Using

$$P^{\mathrm{t}}=\begin{pmatrix} 1 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0\\ g_3 & 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix},$$

we may assume that $\operatorname{Ker}\gamma=\langle e_1^*, e_4^*, e_5^*\rangle$ and we get

Case IIa(ii):

$$\begin{pmatrix} 0 & 0 & x & y & 0\\ 0 & 0 & z & x & y\\ -x & -z & 0 & 0 & 0\\ -y & -x & 0 & 0 & 0\\ 0 & -y & 0 & 0 & 0\end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (x^2-yz, xy, y^2), the ideal of the curvilinear triple point in the plane.

Case IIa(iii). $\mathbb{P}(\operatorname{Ker}\gamma) \cap \mathbb{P}(\langle e_3^*, e_4^*, e_5^* \rangle)$ = a point on $\varphi(\Gamma)$. By Lemma 2.4, we may assume that $Q = e_5^*$ is the point of intersection. Let $\operatorname{Ker}\gamma = \langle e_5^*, e_1^* + g_3 e_3^* + g_4 e_4^*, e_2^* + h_3 e_3^* + h_4 e_4^* \rangle$. Using

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ g_3 & g_4 & 1 & 0 & 0 \\ g_4 & h_4 & 0 & 1 & 0 \\ h_4 & 0 & 0 & 0 & 1 \end{pmatrix},$$

we may assume that $\operatorname{Ker}\gamma = \langle e_5^*, e_1^* - h_4 e_5^*, e_2^* + g e_3^* \rangle = \langle e_5^*, e_1^*, e_2^* + g e_3^* \rangle$, for some g. Hence

$$[\gamma]_E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & g & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$[N]_E = \begin{pmatrix} 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x-gz & y \\ -x & 0 & 0 & z & 0 \\ -y & -x+gz & -z & 0 & 0 \\ 0 & -y & 0 & 0 & 0 \end{pmatrix}.$$

We note that $g = 0$ because otherwise $y = x - gz = 0$ would give another rank 2 map. Hence we get

Case IIa(iii):

$$\begin{pmatrix} 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ -x & 0 & 0 & z & 0 \\ -y & -x & -z & 0 & 0 \\ 0 & -y & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (x^2, xy, y^2, yz), the ideal of a double point on a line (with an irrelevant component).

Case IIa(iv). $\mathbb{P}(\operatorname{Ker}\gamma) \cap \mathbb{P}(\langle e_3^*, e_4^*, e_5^* \rangle)$ = a point outside $\varphi(\Gamma)$. Let $Q = g_3 e_3^* + g_4 e_4^* + g_5 e_5^*$ be the point of intersection.

(a) char $k \neq 2$. We claim that we can make a change of basis so that $Q = e_4^*$. We first assume that $g_4 \neq 0$, say $g_4 = 1$. Switching e_3^* and e_5^* if necessary, we may assume that $g_3 \neq 0$. Let r be a root of $g_5 x^2 - 2x + g_3 = 0$ and let

$$A = \begin{pmatrix} g_3/2r & r \\ g_5/2 & 1 \end{pmatrix}.$$

Then $\det A = (1/2r)(g_3 - g_5 r^2) = (1/2r)(g_3 - r) \neq 0$ because $g_3 = r$ would imply that $g_3(g_3 g_5 - 1) = 0$ and Q becomes a point on $\varphi(\Gamma)$ which is a contradiction. Since

$$S_2(A) = \begin{pmatrix} * & g_3 & * \\ * & 1 & * \\ * & g_5 & * \end{pmatrix},$$

we may assume that $Q = e_4^*$. Now suppose that $g_4 = 0$. Since $Q \in \varphi(\Gamma)$, $g_3 \neq 0$ and $g_5 \neq 0$. Let

$$A = \begin{pmatrix} \sqrt{g_3/2} & \sqrt{g_3/2} \\ -\sqrt{-g_5/2} & \sqrt{-g_5/2} \end{pmatrix}.$$

Then

$$S_2(A) = \begin{pmatrix} * & g_3 & * \\ * & 0 & * \\ * & g_5 & * \end{pmatrix},$$

and we may assume that $Q = e_4^*$ in this case also.

(b) char $k = 2$ and $g_3 = g_5 = 0$. Trivially $Q = e_4^*$ in this case.

(c) char $k = 2$ and $g_3 \neq 0$ or $g_5 \neq 0$. We may assume that $g_5 \neq 0$, say $g_5 = 1$. Let

$$A = \begin{pmatrix} 1 & \sqrt{g_3} \\ 0 & 1 \end{pmatrix}.$$

Since char $k = 2$,

$$S_2(A) = \begin{pmatrix} 1 & 0 & g_3 \\ 0 & 1 & \sqrt{g_3} \\ 0 & 0 & 1 \end{pmatrix},$$

and we may assume that $Q = ge_4^* + e_5^*$ for some g. Since Q is outside $\varphi(\Gamma)$, $g \neq 0$ and

$$A = \begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix}$$

is invertible. Now

$$S_2(A) = \begin{pmatrix} g^2 & 0 & 0 \\ 0 & g & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

and we may assume that $Q = e_4^* + e_5^*$.

Remark. When char $k = 2$ the cases (b) and (c) are different because

$$S_2(A) = \begin{pmatrix} * & 0 & * \\ * & * & * \\ * & 0 & * \end{pmatrix}$$

for all 2×2 invertible A and we cannot find a basis D of V normalizing Γ such that $d_4^* = e_4^* + e_5^*$.

The case $Q = e_4^*$. Let $\operatorname{Ker} \gamma = \langle e_4^*,\ e_1^* + g_3 e_3^* + g_5 e_5^*,\ e_2^* + h_3 e_3^* + h_5 e_5^* \rangle$. Using

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ g_3 & h_3 & 1 & 0 & 0 \\ h_3 & g_5 & 0 & 1 & 0 \\ g_5 & h_5 & 0 & 0 & 1 \end{pmatrix},$$

we may assume that $\operatorname{Ker} \gamma = \langle e_4^*,\ e_1^* - h_3 e_4^*,\ e_2^* - g_5 e_4^* \rangle = \langle e_1^*, e_2^*, e_4^* \rangle$ and we get

Case IIa(iv):

$$\begin{pmatrix} 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ -x & 0 & 0 & 0 & z \\ -y & -x & 0 & 0 & 0 \\ 0 & -y & -z & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is $(x, y) \cap (x, y, z)^2$, the ideal of a single point in the plane with an irrelevant component.

The case $\operatorname{char} k = 2$ *and* $Q = e_4^* + e_5^*$. Let $\operatorname{Ker} \gamma = \langle e_4^* + e_5^*,\ e_1^* + g_3 e_3^* + g_5 e_5^*,\ e_2^* + h_3 e_3^* + h_5 e_5^* \rangle$. Using

$$P^t = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ g & h_3 & 1 & 0 & 0 \\ h_3 & h_3 + g_5 & 0 & 1 & 0 \\ h_3 + g_5 & h_3 + g_5 + h_5 & 0 & 0 & 1 \end{pmatrix},$$

we may assume that

$$\begin{aligned} \operatorname{Ker} \gamma &= \langle e_4^* + e_5^*, e_1^* - h_3(e_4^* + e_5^*), e_2^* - (h_3 + g_5)(e_4^* + e_5^*) \rangle \\ &= \langle e_1^*, e_2^*, e_4^* + e_5^* \rangle \end{aligned}$$

and get

Case IIa(v):

$$\begin{pmatrix} 0 & 0 & x & y & 0 \\ 0 & 0 & 0 & x & y \\ -x & 0 & 0 & z & z \\ -y & -x & -z & 0 & 0 \\ 0 & -y & -z & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is $(x, y) \cap (x, y, z)^2$, the ideal of a single point in the plane with an irrelevant component, as in the previous example.

Case IIb. $(\mathbb{P}(N)_2)_{\text{red}}$ consists of 2 distinct points (2 examples). Again these are distinguished, for example, by the scheme structure of $\mathbb{P}(N)_2$.

Let α and β denote two rank 2 maps. We may assume that $\alpha = e_1 \wedge e_2$ and $\beta = e_3 \wedge e_4$. Let $\gamma = \sum_{1 \leq i < j \leq 5} c_{ij} e_i \wedge e_j$ be a third vector such that α, β, γ forms a basis of N. Subtracting multiples of α and β, we may assume that $c_{12} = 0$ and $c_{34} = 0$. Since N has no nontrivial common kernel, $c_{i5} \neq 0$ for some i. We may assume, after renaming the variables if necessary, that $c_{25} \neq 0$ (if $c_{15} \neq 0$, then switch the first two rows and columns and if

$c_{15}=c_{25}=0$, then switch rows 1 and 2 with 3 and 4), say $c_{25}=1$. Writing e_5 for $c_{23}e_3+c_{24}e_4+c_{25}e_5$, we may assume that N is of the form

$$\begin{pmatrix} 0 & x & * & * & * \\ -x & 0 & 0 & 0 & z \\ * & 0 & 0 & y & * \\ * & 0 & -y & 0 & * \\ * & -z & * & * & 0 \end{pmatrix},$$

where $*$ denotes a multiple of z.

Using row 2, we may assume that the $(1,5)$-entry is 0. Since $x=y=0$ is to define a rank 4 map, we may assume that $(1,4)$-entry is z and $(1,3)$-entry is 0. We have

$$\begin{pmatrix} 0 & x & 0 & z & 0 \\ -x & 0 & 0 & 0 & z \\ 0 & 0 & 0 & y & sz \\ -z & 0 & -y & 0 & tz \\ 0 & -z & -sz & -tz & 0 \end{pmatrix}.$$

Case IIb(i): $s=0$. In this case $t=0$ or else $y=tx-z=0$ gives a third rank 2 map.

Case IIb(i):

$$\begin{pmatrix} 0 & x & 0 & z & 0 \\ -x & 0 & 0 & 0 & z \\ 0 & 0 & 0 & y & 0 \\ -z & 0 & -y & 0 & 0 \\ 0 & -z & 0 & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (xy, yz, z^2), the ideal of one reduced and one double point.

Case IIb(ii): $s\neq 0$. We may first use the row 3 to assume that $t=0$. We next multiply the row 3 by $(1/s)$ and write y for $(1/s)\,y$ to get

Case IIb(ii):

$$\begin{pmatrix} 0 & x & 0 & z & 0 \\ -x & 0 & 0 & 0 & z \\ 0 & 0 & 0 & y & z \\ -z & 0 & -y & 0 & 0 \\ 0 & -z & -z & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (xy, yz, z^2, xz), the ideal of two reduced points (with an irrelevant component).

Case III. Here $\mathbb{P}(N)_2$ is set theoretically a line in the Grassmannian of lines in $\mathbb{P}(V) \cong \mathbb{P}^4$ and we see from Lemma 1.2(i) that there exist a point P and a 2-plane U of V such that $P \subset \mathbb{P}(N)_2 \subset U$. These correspond to a hyperplane L of V^* which contains the kernel of every map in N_2 and a 2-plane L' of V^* which is the intersection of those kernels.

Case III(i). There is an element $\gamma \in N - N_2$ such that $\operatorname{Ker} \gamma \not\subset L$. Let α and β be two distinct elements of N_2. We choose a basis of V^* in such a way that the first vector is from $\operatorname{Ker} \gamma$, the last two from L', the second from $\operatorname{Ker} \beta$, and the third from $\operatorname{Ker} \alpha$. If we choose α, β, and γ as a basis of N, then we can represent N as

$$\begin{pmatrix} 0 & x & y & 0 & 0 \\ -x & 0 & * & * & * \\ -y & * & 0 & * & * \\ 0 & * & * & 0 & * \\ 0 & * & * & * & 0 \end{pmatrix},$$

where $*$ denotes a multiple of z.

If γ (in fact γ restricted to L), viewed as a skew-form on L, is nonsingular on L', then we get

Case III(i):

$$\begin{pmatrix} 0 & x & y & 0 & 0 \\ -x & 0 & z & 0 & 0 \\ -y & -z & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & -z & 0 \end{pmatrix}.$$

If the form is 0 on L', then any basis is dual to a basis of L/L', which we may lift to (any) L'' complementary to L' in L—so we get

Case III(ii):

$$\begin{pmatrix} 0 & x & y & 0 & 0 \\ -x & 0 & 0 & z & 0 \\ -y & 0 & 0 & 0 & z \\ 0 & -z & 0 & 0 & 0 \\ 0 & 0 & -z & 0 & 0 \end{pmatrix}.$$

Case III(ii). $\operatorname{Ker}\gamma \subset L$ for all $\gamma \in N$. Pick any $\gamma \in N - N_2$. We first note that $\operatorname{Ker}\gamma$ can not be contained in L'-else an element of L' is in the kernel of every $n \in N$. Let α be the element of N_2 such that $\operatorname{Ker}\alpha = L' + \operatorname{Ker}\gamma$. Let β be an element of N_2 distinct from α. If we choose a basis of V^* in such a way that the last two vectors are from L', the third vector is from $\operatorname{Ker}\alpha$, and the second vector is from $\operatorname{Ker}\beta$, then we can represent N as

$$\begin{pmatrix} 0 & x+cz & y & * & * \\ -x-cz & 0 & 0 & * & * \\ -y & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & * \\ * & * & 0 & * & 0 \end{pmatrix},$$

where $*$ denotes a multiple of z.

The $(1,4)$- and $(1,5)$-entries cannot be both zero because otherwise $x = -c$, $y = 0$, and $z = 1$ gives a map in N whose kernel is not contained in L. We can therefore assume that N can be represented as

$$\begin{pmatrix} 0 & x & y & 0 & z \\ -x & 0 & 0 & * & * \\ -y & 0 & 0 & 0 & 0 \\ 0 & * & 0 & 0 & * \\ -z & * & 0 & * & 0 \end{pmatrix},$$

where $*$ denotes a multiple of z.

Since $x = y = 0$ defines a rank 4 map, we may assume that the $(2,4)$-entry is z after multiplying the row 4 and the column 4 by a suitable constant. We now use the column 4 to make the $(2,5)$-entry 0. Now N is in the form

$$\begin{pmatrix} 0 & x & y & 0 & z \\ -x & 0 & 0 & z & 0 \\ -y & 0 & 0 & 0 & 0 \\ 0 & -z & 0 & 0 & cz \\ -z & 0 & 0 & -cz & 0 \end{pmatrix}.$$

Since the pfaffians are $(0, cyz, cxz + z^2, 0, yz)$ and their common zero set is $z = 0$, $c = 0$-else $x = -1/c$, $y = 0$, and $z = 1$ is in the zero set.

Case III(iii):

$$\begin{pmatrix} 0 & x & y & 0 & z \\ -x & 0 & 0 & z & 0 \\ -y & 0 & 0 & 0 & 0 \\ 0 & -z & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Case IV. Let α, β, γ be three elements of N_2 which span N. By our non-degeneracy hypothesis, the corresponding lines must together span $\mathbb{P}(V)$. There are up to automorphisms of $\mathbb{P}(V)$ only 2 such configurations:

Case IV(i). There is a pair of intersecting lines.
We may write $\alpha = e_1 \wedge e_2$ and $\beta = e_1 \wedge e_3$. The third line can't meet the plane spanned by α and β (else they do not span $\mathbb{P}^4$.) Hence we can write the third as $\gamma = e_4 \wedge e_5$.

Case IV(i):

$$\begin{pmatrix} 0 & x & y & 0 & 0 \\ -x & 0 & 0 & 0 & 0 \\ -y & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & z \\ 0 & 0 & 0 & -z & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is $(x, y) \cap (z)$, the ideal of a point and a line.

Case IV(ii). No two of the three lines intersect. We can choose the first two to be $\alpha = e_1 \wedge e_2$ and $\beta = e_3 \wedge e_4$. We can write the third as $\gamma = (c_1 e_1 + c_2 e_2 + c_3 e_3 + c_4 e_4) \wedge e_5$. Since γ does not meet α or β, we may assume that $c_1 = c_3 = 1$. Writing e_1 for $e_1 + c_2 e_2$ and e_3 for $e_3 + c_4 e_4$, we may assume that $\gamma = (e_1 + e_3) \wedge e_5$.

Case IV(ii):

$$\begin{pmatrix} 0 & x & 0 & 0 & z \\ -x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y & z \\ 0 & 0 & -y & 0 & 0 \\ -z & 0 & -z & 0 & 0 \end{pmatrix}.$$

An easy computation shows that the ideal of pfaffians is (xy, xz, yz), the ideal of three linearly independent reduced points.

3. THE LINEAR SYZYGY CONJECTURES IN THE CASE $k=3$, $w=5$

We will assume that the reader of this section is familiar with the definitions and notation of our previous paper. The classification carried out in the previous section allows us to decide whether the linear syzygy conjectures hold in the case $k=3$, $w=5$ (that is, the critical case for 3-generated modules over a polynomial ring in 5 variables) by checking them on modules determined by 3-dimensional subspaces of $\Lambda^3 W$, where W is a 5-dimensional space. If we set $V := W^*$, then $\Lambda^3 W = \Lambda^3 V^*$, so we may use the classification from Section 2, and we must check the result for the each of the 13 modules corresponding to the 13 cases of the classification. This is, of course, a job for a machine, and we were able to carry out the computations using the computer algebra program Macaulay of Bayer and Stillman [2]. The results verify all our conjectures in this case. We computed a number of invariants of the situation, and some of the results are given in the two tables below, which we will now describe.

Table I lists the 13 modules themselves, with some of their properties: the first column names them in a way corresponding to the classification of the last section (but we have given them in a different order, grouping them according to the betti numbers of the free resolution of the module $\mathcal{M}$). The second column gives the generators of $M^* \subset \Lambda^3 W = \Lambda^2 V$ according to the following scheme: assuming that we have chosen a basis

$$e_1, e_2, ..., e_5$$

of $V = W^*$, we replace $e_i \wedge e_j$ by ij. Thus, for example, the first row of the table corresponds to the situation where M^* is generated by

$$e_1 \wedge e_5 + e_2 \wedge e_4 + e_3 \wedge e_5$$

$$e_1 \wedge e_4 + e_2 \wedge e_5$$

$$e_2 \wedge e_3 + e_4 \wedge e_5.$$

The third column of the table gives a minimal presentation matrix of $\mathcal{M}$ as a module over $k[W] = k[a, b, c, d, e]$, where a, b, c, d, e are a basis of W dual to the basis $e_1, ..., e_5$ of V; the columns represent the relations. The fourth column gives the "graded betti numbers of $\mathcal{M}$— that is, the ranks and degrees of generators of the free modules in a minimal free resolution. Perhaps an example will best explain its meaning: the diagram

3	10	5	1	—	—
—	—	21	35	22	5

TABLE I

Name	M* in wedge W	Presenting matrix	Betti numbers of M	Annihilator of M	dim/deg of M
General	14+23 15+24 25+34	0 0 -e d -e 0 -c -c -b a 0 -e 0 -c d c 0 b a 0 e d c b 0 0 a 0 0 0	3 10 5 1 - - - - 21 35 22 5	square of the maximal ideal	0/8
1 point (i)	13+24 14+25 12	-e 0 c 0 -e -d 0 -a b d c 0 -e 0 c -a b 0 0 0 0 0 0 0 -e -d -c	3 9 9 3	6 quadrics and one cubic	2/3
1 point (ii)	13+24 14+25 23	-e 0 c 0 -e -d 0 -a b d c 0 -e 0 c -a b 0 0 -e -d 0 0 0 0 0 -a	3 9 9 3	6 quadrics and one cubic	2/3
2 points (i)	12 34 14+25	0 0 0 0 0 0 -e -d -c -e 0 0 -b 0 -a 0 0 0 0 d c 0 -e c -a b 0	3 9 9 3	6 quadrics and 1 cubic	2/3
3 spanning points	12 34 15+35	0 0 0 0 0 0 -e -d -c -e 0 -b 0 0 -a 0 0 0 d -b 0 d -a+c 0 b 0 0	3 9 9 3	6 quadrics and 1 cubic	2/3
1 point (iii)	13+24 14+25 34	0 -e 0 c 0 -e -d 0 -a b 0 d c 0 -e 0 c -a b 0 -e 0 0 -b 0 0 -a 0 0 0	3 10 11 5 1 - - - 3 6 4 1	(e,ac+bd)+ (a,b)^2	2/2

2 points	12	0 0 0 0 0 0 0 -e -d -c	3 10 11 5 1 -	(e)+
(ii)	34	-e 0 0 -b 0 0 -a 0 0 0	- - 3 6 4 1	(a,b)(c,d)
	14+25+35	d d -b+c 0 -e -a c -a b 0		
1 point	13+24	0 -e 0 c 0 -e -d 0 -a b	3 10 8 1 - -	(a,b,c,d,e)
(iv)	14+25	0 d c 0 -e 0 c -a b 0	- - 10 20 13 3	(a,b,d)
	35	d 0 -b 0 0 -a 0 0 0 0		
1 point	13+24	0 e 0 c 0 e d 0 a b		
char. 2	14+25	0 d c 0 e 0 c a b 0	Same as ''1 point (iv)''	
	34+35	d+e 0 b b 0 a a 0 0 0		
line	12	0 0 0 0 0 0 0 d e c	3 10 14 11 5 1	
(i)	13	0 0 0 0 0 d e 0 0 -b		(d,e)
	23+45	a b c d e 0 0 0 0 a		
line	12	0 0 0 0 0 e 0 d -c	3 9 10 5 1	(b,c,d,e)
(ii)	13	0 0 0 0 d 0 e 0 b		(d,e)
	24+35	b c d e 0 0 a a 0		
line	12	0 0 0 0 0 -e -d -c	3 8 7 2	(bd+ae)+
(iii)	13	0 0 0 -e -d 0 0 b		(e)(c,d,e)+
	15+24	-e c d c 0 b -a 0		(d)(c,d)
line	12	0 0 0 0 0 -e -d -c	3 8 7 2	(d,e)(a,b,c)
and point	13	0 0 0 -e -d 0 0 b		
	45	-c -b -a 0 0 0 0 0		

denotes a resolution of the form

$$\begin{array}{llllll} S^3 \leftarrow S(-1)^{10} \leftarrow & S(-2)^5 \leftarrow & S(-3) & & & \\ & \oplus \quad \nwarrow & \oplus \quad \nwarrow & & & \\ & S(-4)^{21} \leftarrow & S(-5)^{35} \leftarrow & S(-6)^{22} \leftarrow & S(-7)^5 \leftarrow 0. & \end{array}$$

The fifth column gives some information about the annihilator of $\mathcal{M}$. The reduced supports of the modules that occur turn out to be quite simple projective varieties, and it would be interesting to study the relation between the examples and their geometry. The final column gives the Krull dimension and degree (that is, multiplicity) of $\mathcal{M}$ in the form x/y, where x is the dimension and y is the degree.

Table II gives information about the various loci mentioned in our conjectures. Each box potentially contains two pairs of numbers,

$$u/v$$
$$[x/y]$$

the upper one representing the dimension/degree of a given locus, and the lower one representing the dimension/degree of the part of that locus corresponding to the "pure vectors." We have suppressed the lower entry where it agrees with the upper one.

In the first column, rank0 ($=\operatorname{Ker}\varphi$, in the notation of our previous paper) refers to the locus in $\Lambda^2 W^*$ of vectors that map to 0 in the module of linear relations, R, and [pure0] ($=\Gamma_0$ in the notation of our previous paper) refers to the intersection of rank0 with Γ, the cone of pure vectors. The upper numbers give the dimension in each case (the locus rank 0 is of course a linear space) and the lower pairs of numbers, in brackets, give the dimension and degree of pure0 in case it is nonzero. Similarly, in the second column, the upper pair of numbers refers to the dimension and degree of the rank 1 locus in $\Lambda^2 W^*$ and the lower pair of numbers, in brackets, give the dimension and degree of Γ_1, the intersection of the rank 1 locus with the set of pure vectors.

In the third column, the two pairs of numbers give the dimension and degree of the images of these loci in the space of linear relations R ($R1 = R_1$ and pure$R1 = \varphi(\Gamma_1)$ in the notation of our previous paper). The conclusion of the linear syzygy conjecture (resp. strong linear syzygy conjecture) says that the dimension of $R1$ (resp. pure$R1$) is at least 3. The conclusion of the generic injectivity conjecture says that the dimension pure1 in the second column is the same as the dimension of pure$R1$ in the third column.

In the fourth and fifth columns we give the numbers relevant to the epimorphism and image conjectures: $M1$ refers to the set of vectors in M which are annihilated by a nonzero linear form, while pure$M1$ refers to the

TABLE II

Name	rank0 [pure0]	rank1 [pure1]	R1 [pureR1]	M1 [pureM1]	W1 [pureW1]
General	0/1	3/9	3/9	3/1	3/4
1 point (i)	1/1	5/3 [4/4]	4/3 [3/4]	2/2	3/1 [2/6]
1 point (ii)	1/1	4/3 [3/9]	3/3	2/3	3/3
2 points (i)	1/1	4/3 [3/9]	3/3	2/3	3/3
3 spanning points	1/1	4/3 [3/9]	3/3	2/3	3/3
1 point (iii)	0/1	4/1 [3/9]	4/1 [3/9]	3/1	4/1 [3/4]
2 points (ii)	0/1	4/1 [3/9]	4/1 [3/9]	3/1	4/1 [3/4]
1 point (iv)	0/1	3/9	3/9	3/1	3/4
1 point char 2	Same as ''1 point (iv)''				
line (i)	0/1	5/1 [4/4]	5/1 [4/4]	3/1	5/1 [4/1]
line (ii)	1/1	5/1 [4/4]	4/1	3/1	4/1
line (iii)	2/1	5/3 [4/4]	3/3 [3/2]	2/1	3/1 [2/4]
line and point	2/1	5/3 [4/4]	3/3	2/1	3/1

vectors annihilated by a linear form in a relation coming from an element of Γ_1. $W1$ refers to the set of linear forms annihilating some element of M, and pure$W1$ refers to the linear forms annihilating an element of M because of a relation coming from Γ_1. The conclusion of the Image Conjecture (resp. Epimorphism Conjecture) is that the dimension of pure$M1$ (resp. pure$W1$) in the fourth (resp. fifth) column is at least 3.)

Remarks: (1) In each of the six examples 1 point (i), 1 point (ii), 2 points (i), 3 spanning points, line (iii), and line and point, there is an element that is killed by 3 linearly independent linear forms. The three conjectures, Generic Injectivity Conjecture, Image Conjecture, and Epimorphism Conjecture, deal with "nondegenerate" cases and above examples violate the hypotheses of these conjectures.

(2) In the examples

1 point (i)

1 point (ii)

2 points (i)

3 spanning points

the module M has the same resolution in each case. These four examples belong together in several senses:

(a) In each case the annihilator of $\mathcal{M}$ defines a scheme X in $\mathbb{P}(W^*) = \mathbb{P}^4$ which is purely 1-dimensional, of degree 3, and of arithmetic genus -2: in the last case this scheme consists of 3 disjoint reduced lines, in the second to the last case a double line and a disjoint reduced line, and in the first two cases triple lines which seem to be degenerations of the other two.

(b) In each case we may recover $\mathcal{M}$ as a graded module from X in the form

$$\mathcal{M} = \sum_{p=-\infty}^{\infty} H^0(X, \mathcal{O}_X(p)).$$

It would be nice to have similarly geometric descriptions of the other modules. Of course, it may well be that we are looking, in some cases at least, only at the linear part of the presentation of the module, and that in order to see what is going on one would have to adjoin relations of higher degree.

It would seem that the underlying reason for these relations is that the examples are probably all flat specializations of the "3 spanning points" (this is obvious for the second and third, but not quite obvious for the first, and we have not tried to prove it).

(3) The two examples "1 point (iii)" and "2 points (ii)" seem similarly related by degeneration (though again we have not studied this.) The support of $\mathcal{M}$ in "2 points (ii)" is a pair of skew lines, while the support of $\mathcal{M}$ in "1 point (iii)" is a double line on a smooth quadric surface.

(4) All the examples given in Tables I and II are distinguished from one another by the invariants given (including the scheme-theoretic support of M, which is described above in the cases where it is not given explicitly in Tables I and II) *except* the pair of examples "1 point (iv)" and "1 point char 2," which have the same invariants, even in characteristic 2.

4. Macaulay Programs for the Linear Syzygy Conjecture

We include one example of the output of the program we used in our computation. The program consists of two scripts:

''ls_prep_11_89'' = "linear syzygy preparation"

and

''ls_11_89'' = "linear syzygy."

These scripts use several scripts which are part of the standard Macaulay package (the references section below tells how to get Macaulay and the standard scripts.)

We give a sample run, this one dealing with the case "one point i," corresponding to the skew symmetric matrix printed at the beginning of the run. The file casepoint.1, which is listed as a parameter on the command line of ls_11_89, is a data file containing the description of this skew symmetric matrix; it is printed out at the end.

```
Macaulay version 3.0, created 8/14/89
;The following takes about 56 seconds on a Mac II with 50
Mhz accelerator:

%<ls_prep_11_89
% <ls_11_89 casepoint.1
; 0  z  x y 0
; -z 0  0 x y
; -x 0  0 0 0
; -y -x 0 0 0
; 0  -y 0 0 0
```

```
*************** The pfaffians: ****************
the ideal of pfaffians
; x2 xy y2
; codimension : 2
; degree      : 3
the radical of the ideal of pfaffians
XXXXXXXXXXXXXXXXXXXXXXXXX
Pure dimensional component:
; codimension : 2
; degree      : 3
; x2 xy y2
and its radical:
; codimension : 2
; degree      : 1
; x y
XXXXXXXXXXXXXXXXXXXXXXXXX
; x y
*************** The module: ****************
Presentation matrix, codim, and degree:
; -e 0 c 0 e d 0 a  -b
; d  c 0 e 0 0 a -b 0
; 0  0 0 0 0 0 e d  c
; codimension : 3
; degree      : 3
; total:     3    9    9    3
; -------------------------------
;    0 :     3    9    9    3
Its annihilator:
; c2 cd ce d2 de e2 a2c+abd+b2e
; codimension : 3
; degree      : 3
; total:     1    7   11    6    1
; ------------------------------------
;    0:      1    -    -    -    -
;    1:      -    6    8    3    -
;    2:      -    1    3    3    1
XXXXXXXXXXXXXXXXXXXXXXXXX
Pure dimensional component:
; codimension : 3
; degree      : 3
; c2 cd ce d2 de e2 a2c+abd+b2e
```

```
and its radical:
; codimension : 3
; degree      : 1
; c d e
XXXXXXXXXXXXXXXXXXXXXXXX
********** In 2 W* **********
rank0
; codimension : 9
; degree      : 1
pure0
; codimension : 9
; degree      : 1
rank1
; codimension : 5
; degree      : 3
pure1
; codimension : 6
; degree      : 4
********** In M tensor W ************
R in 15 space:
; codimension : 6
rank1R in 15 space
; codimension : 11
; degree      : 3
pure1R in 15 space
; codimension : 12
; degree      : 4
************ in M *******************
image of rank 1 in M
; codimension : 1
; degree      : 2
image of pure1 in M
; codimension : 1
; degree      : 2
************ in W *******************
image of rank1 in W
; codimension : 2
; degree      : 1
image of pure1 in W
; codimension : 3
; degree      : 6
*******************************************
*******************************************
```

```
;
;The contents of the file casepoint.1:
;


5
5
0
-z
-x
-y
0
z
0
0
-x
-y
x
0
0
0
0
y
x
0
0
0
0
y
0
0
0
```

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