Syzygy ideals for determinantal ideals and the syzygetic Castelnuovo lemma

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If $I$ is an ideal in a (local, or graded) ring $S$ and $s$ is a generator of the module of $i$th syzygies of $I$, then the syzygy ideal of $s$, recently studied by Schreyer and others (see Ehbauer [1994]), is roughly speaking the smallest ideal $I'$ inside $I$ such that $s$ is the image of an $i$th syzygy of $I'$. The syzygy ideal can be defined in terms of the Ext$(k,k)$-module structure on Tor$(S/I,k)$, where $k$ is the residue field. In this note we will give an explicit computation of this module structure, and thus of the syzygy ideals, in the case where $I$ is generated by the maximal minors of an $f \times g$ matrix $\phi$ of linear forms over a polynomial ring, and $I$ has “expected” codimension $f - g + 1$.

In Section 3 we conjecture that the behavior of syzygy ideals characterizes the 1-genericity (Eisenbud [1988]) of $\phi$ in the sense that $\phi$ is 1-generic if and only if the syzygy ideal of each $(f-g)^{th}$ syzygy of $I$ is all of $I$. We prove the “if” statement of this conjecture, and we prove the “only if” statement in the case where $g = 2$. As an application we give a new, direct proof of a scheme-theoretic generalization of Green’s “syzygetic Castelnuovo Lemma” (Green [1984]) proved independently by Yanagawa [1994] and Ehbauer [1994]: A finite subscheme $\Gamma \subset \mathbb{P}^r$ of length $\geq r + 3$ lies on a smooth rational normal curve iff $\Gamma$ contains a subscheme of length $r + 3$ in linearly general position, and Tor$_{r-2}(I_{\Gamma}, k)_r \neq 0$.

Here is a typical example in which syzygy ideals arise. Let $C$ be a smooth curve of genus $g$, canonically embedded in $\mathbb{P}^{g-1}$. Let $S = k[x_0, \ldots, x_{g-1}]$ be the homogeneous coordinate ring of $\mathbb{P}^{g-1}$, and let $I$ be the homogeneous ideal of $C$. It is known (see Saint-Donat [1973]) that if Tor$_{g-4}(I, k)_g \neq 0$, then the curve $C$ is trigonal and lies on a 2-dimensional rational normal scroll, whose ideal $J$ satisfies

$$\text{Tor}_{g-4}(J, k)_g = \text{Tor}_{g-4}(I, k)_g.$$
A consequence of Theorem 3.1 below is that \( J \) is the syzygy ideal of any syzygy \( \tau \in \text{Tor}_{g-4}(I, k)_{g-2} \).

We are grateful to Frank-Olaf Schreyer for introducing us to syzygy varieties and their applications.

1 Syzygy submodules and ideals

To begin, we give a definition of syzygy ideals somewhat more general than the one given in Ehhauer [1994]. Suppose that \( k \) is a field, \( S = k[x_0, \ldots, x_n] \) is a polynomial ring over \( k \), and \( m \) is the ideal generated by the variables of \( S \). Let \( M \) be a finitely generated graded \( S \)-module, perhaps an ideal of \( S \). The graded vector space \( \text{Tor}_i(M, k) \) may be identified with the space of generators of the \( i \)-th free module in the minimal free resolution of \( M \), and we refer to elements of \( \text{Tor}_i(M, k) \) as \( i \)-th syzygies of \( M \). Let \( \tau \in \text{Tor}_i(M, k) \) be a syzygy of degree \( j \). The image in \( M \otimes k \) of \( \tau \otimes \text{Ext}_i(k, k) \) under the multiplication map

\[
\text{Tor}_i(M, k) \otimes \text{Ext}_i(k, k) \longrightarrow \text{Tor}_0(M, k) = M \otimes k
\]

is a subspace denoted in the sequel \( \tau \text{Ext}_i(k, k) \). There is a natural identification \( \text{Ext}_i^*(k, k) = \wedge^* W^* \), where \( W := S_1 \) denotes the vector space of linear forms in \( S \), and \( W^* \) is its \( k \)-dual. Thus \( \text{Ext}_i^*(k, k) \) is concentrated in degree \( i \) and \( \tau \text{Ext}_i(k, k) \subset (M \otimes k)_{j-i} \). Lifting this space to a space in \( M_{j-i} \), we get a subspace that is well defined modulo the generators of \( M \) of lower degree. Therefore, by adding \( M_{j-i} \) we get a well defined module of \( M \).

**Definition** The syzygy submodule of \( \tau \) is the module generated by representatives of \( \tau \text{Ext}_i(k, k) \) together with \( M_{j-i} \).

The syzygy submodule was defined in Ehhauer [1994] only in the case where \( M \) is an ideal containing no linear forms in the polynomial ring \( S \), and \( \tau \) has degree \( i + 2 \): Let \( K^* \) be the Koszul complex of \( S \). We can identify \( \text{Tor}_i(M, k)_{i+2} \) with \( \text{Tor}_{i+1}(S/M, k)_{i+2} \), and thus with the Koszul homology \( (H_{i+1}(S/M \otimes K^*))_{i+2} \). The element \( \tau \) is represented by a cycle \( \tau' \in \wedge^{i+1} W \otimes (S/M)_1 = \wedge^{i+1} W \otimes M_2 \), where now the \( \otimes \) is taken over \( k \). Since \( \tau' \) is a cycle, its image in \( \wedge^i W \otimes (S/M)_2 \) is 0, and thus its image \( d\tau' \) under the differential \( d \) of \( K \) lies in \( \wedge^i W \otimes M_2 \subset \wedge^i W \otimes S_2 \). We may regard \( d\tau' \) as a map \( d\tau' : \wedge^i W^* \longrightarrow M_2 \); the image of this map is the syzygy submodule of \( \tau \) described above. The same method works whenever \( M \) is a submodule of a free \( S \)-module.

The pairing between \( \text{Ext}_i^*(k, k) \) and \( \text{Tor}_i^*(M, k) \) can be understood directly from the Yoneda description of \( \text{Ext} \). Given an extension

\[
e : \quad 0 \longrightarrow k \longrightarrow A_j \longrightarrow A_{j-1} \longrightarrow \cdots \longrightarrow k \longrightarrow 0,
\]
we break it into a succession of short exact sequences

\[ 0 \longrightarrow B_{i+1} \longrightarrow A_i \longrightarrow B_i \longrightarrow 0, \]

with \( B_i = B_{i+1} = k \). Each of these sequences induces a long exact sequence in \( \text{Tor}_i(M, -) \) and in particular a connecting homomorphism

\[ \text{Tor}_{i-1}(M, B_i) \longrightarrow \text{Tor}_{i-1}(M, B_{i+1}). \]

The composition of these maps is the map

\[ \text{Tor}_i(M, k) \longrightarrow \text{Tor}_{i-1}(M, k) \]

that is multiplication by \( \epsilon \). To actually compute this map, we use the description of \( \text{Ext}^i(k, k) \) via the Koszul complex \( K^* \), which is a free resolution of \( k \), to represent an element \( \epsilon \in \text{Ext}^i(k, k) \) as a map \( \wedge^j W \longrightarrow k \). Let

\[ \ldots \longrightarrow G_i \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \]

be a minimal free resolution of \( M \). An \( i \)th syzygy \( \tau \) of \( M \) may be represented by an element of \( G_i \), and thus by a map \( \tilde{\tau} : S \longrightarrow G_i \). The image of \( G_i \) is in \( mG_i \), and thus the composite \( G_i \longrightarrow G_i \longrightarrow S \) lifts along the first map of the Koszul complex to a map \( G_i \longrightarrow \wedge^j W \otimes S \). Continuing the lifting we get a commutative diagram

\[
\begin{array}{ccc}
G_i & \longrightarrow & G_i \\
\text{lift} \downarrow & & \text{lift} \downarrow \\
S & \longrightarrow & \wedge^j W \otimes S
\end{array}
\]

Dualizing \( \tilde{\tau}_j \), tensoring with \( k \), and composing with \( \epsilon^* \), we get a composite map

\[ k \longrightarrow \wedge^j W^* \longrightarrow G_{i-j} \otimes k = \text{Tor}_{i-j}(M, k). \]

The image of \( 1 \in k \) is the class \( \tau \epsilon \).

If we work with \( G_i \otimes K^* \) in place of \( K^* \) we may do this for all the \( i \)th syzygies at once and we obtain:

**Lemma 1.1** Let \( \ldots \longrightarrow G_i \longrightarrow \ldots \longrightarrow G_0 \longrightarrow M \longrightarrow 0 \) be a minimal free resolution of \( M \). If

\[
\begin{array}{ccc}
G_i^* & \longrightarrow & G_i^* \\
m_{i,0} \downarrow & & m_{i,1} \downarrow \\
G_i^* \otimes S & \longrightarrow & G_i^* \otimes W \\
\end{array}
\]

then

\[
\begin{array}{ccc}
G_i^* & \longrightarrow & G_i^* \\
m_{i,0} \downarrow & & m_{i,1} \downarrow \\
G_i^* \otimes S & \longrightarrow & G_i^* \otimes W \\
\end{array}
\]
is a lifting of the natural identification $m_{i,0} : G_i^* \to G_i^* \otimes S$, then the multiplication map

$$\mu_{i,j} : \text{Tor}_i(M, k) \otimes \text{Ext}^j(k, k) \to \text{Tor}_{i-j}(M, k)$$

is the composition

$$\text{Tor}_i(M, k) \otimes \text{Ext}^j(k, k) = G_i \otimes \Lambda^j W^* \xrightarrow{m_{i,j} \otimes k} G_{i-j} \otimes k$$

Below we shall carry this computation through in general when $M$ is a determinantal ideal and the resolution is the Eagon-Northcott complex. Here is a simple special case:

**Example 1.2** Let $I$ be the ideal of three points in the plane, $(xy, xz, yz) \subset S := k[x, y, z]$. The ideal $I$ has free resolution

$$0 \longrightarrow S^2(-3) \xrightarrow{\phi} S^3(-2) \longrightarrow I \longrightarrow 0.$$ 

Choosing bases $e_1, e_2$ for $S^2(-3)$ and $f_1, f_2, f_3$ for $S^3(-2)$ we may write $\phi$ as a matrix

$$\phi = \begin{pmatrix} z & 0 \\ -y & -y \\ 0 & x \end{pmatrix}.$$ 

Note that $I$ is the ideal of $2 \times 2$ minors of $\phi$, and the resolution is the Eagon-Northcott complex. From the commutativity of the diagram

$$\begin{array}{cccc}
S^2(-3) & \xrightarrow{\phi} & S^3(-2) & \xrightarrow{(xy \ xz \ yz)} S \\
& & & & \\
(1) & & & & \\
0 & & & & \\
0 & & & & \\
S(-3) & \xrightarrow{(0 \ 0 \ 0 \ 1 \ 0 \ 0 \ 0 \ 1)} & S(-2) \otimes W^* \\
\end{array}$$

and the computation above we see that the syzygy ideal of $e_1$ is the image of the composite map

$$(xy \ xz \ yz) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = (0 \ xz \ yz).$$

Thus the syzygy ideal of $e_1$ is the ideal $(xz, yz)$ corresponding to a line and
a point:

\[
\begin{array}{c}
\bullet \\
\bullet \\
\bullet 
\end{array}
\]

The dots shown are the three points defined by the ideal \(I\).

Similarly, the syzygy ideal of the second generator is \((xz, yz)\); the syzygy ideal of the difference of these two generators is \((xy, yz)\); and the syzygy ideal of the sum (or for any other linear combination) is \(I\) itself. We shall see that this variety is possible only because of the zeros in the matrix \(\phi\).

2 The module structure of the Eagon-Northcott complex

Recall (for example from Eisenbud [1995, Appendix 2.6]) that if \(F, G\) are free modules of ranks \(f \geq g\) over \(S\) and \(\phi: F \rightarrow G\) is a map whose ideal \(I := I_\phi(\phi)\) of \(g \times g\)-minors has the "expected" codimension \(f - g + 1\), then the free resolution of \(I\) is given by the Eagon-Northcott complex

\[
0 \longrightarrow (\text{Sym}_{f-j}G)^* \otimes \wedge^j F \longrightarrow \cdots \longrightarrow G^* \otimes \wedge^{g+1} F \longrightarrow \wedge^g F \longrightarrow I \longrightarrow 0.
\]

Henceforward we suppose that \(\phi\) is represented by a matrix of linear forms. In more invariant terms, we can think of \(\phi\) as a map of vector spaces \(\phi' : \bar{F} \otimes \bar{G}^* \longrightarrow \bar{S}_1 = W\), where \(\bar{X}\) denotes \(k \otimes_S X\) and all other tensor products are taken over \(k\).

We will explicitly compute the multiplication maps

\[
\mu_{i,j} : \text{Tor}_i(I, k) \otimes \text{Ext}^j(k, k) \longrightarrow \text{Tor}_{i-j}(I, k).
\]

To make this explicit we identify \(\text{Tor}_i(I, k) = (\text{Sym}_i \bar{G})^* \otimes \wedge^{g+i} \bar{F}\), and \(\text{Ext}^j(k, k) = \wedge^j W^*\).

To describe the \(\mu_{i,j}\) we use two additional multilinear constructions: First, the Cauchy decomposition of \(\wedge^j (\bar{F}^* \otimes \bar{G})\) (see for example Fulton-Harris [1991]) yields maps \(c_j : \wedge^j (\bar{F}^* \otimes \bar{G}) \longrightarrow (\wedge^j \bar{F}^*) \otimes (\text{Sym}_j \bar{G})\), that can be conveniently defined by the formula

\[
c_j : (f_1 \otimes g_1) \wedge \cdots \wedge (f_j \otimes g_j) \mapsto (f_1 \wedge \cdots \wedge f_j) \otimes (g_1 \cdots g_j).
\]

Second, we use the module structures (Boubakese: inner products)

\[
m : \wedge^j \bar{F}^* \otimes \wedge^{g+i} \bar{F} \longrightarrow \wedge^{g+i-j} \bar{F}
\]

\[
n : (\text{Sym}_i \bar{G})^* \otimes \text{Sym}_j \bar{G} \longrightarrow (\text{Sym}_{i-j} \bar{G})^*
\]
For these see for example Eisenbud [1995, Appendix 2.4].

**Theorem 2.1** With notation and identifications as above, the map

\[ \mu_{i,j} : \text{Tor}_i(I, k) \otimes \text{Ext}_j^i(k, k) \longrightarrow \text{Tor}_{i-j}(I, k) \]

is the composite

\[
\begin{align*}
(\text{Sym}_i \ddot{G})^* \otimes \wedge^g F \otimes \wedge^j W & \xrightarrow{1 \otimes 1 \otimes \phi'} (\text{Sym}_i \ddot{G})^* \otimes \wedge^{g+i} \ddot{F} \otimes \wedge^j (\ddot{F}^* \otimes \ddot{G}) \\
& \xrightarrow{1 \otimes 1 \otimes \phi} (\text{Sym}_i \ddot{G})^* \otimes \wedge^{g+i} \ddot{F} \otimes \wedge^j \ddot{F}^* \otimes \text{Sym}_j \ddot{G} \\
& \xrightarrow{n \otimes m} (\text{Sym}_{i-j} \ddot{G})^* \otimes \wedge^{g+i-j} \ddot{F}.
\end{align*}
\]

**Proof** Let \( \ddot{m}_{i,j} \) be the composite map defined in the statement of Theorem 2.1, and let \( m_{i,j} \) be the map obtained by tensoring \( \ddot{m}_{i,j} \) with \( S \). By Lemma 1.1 it suffices to show that, for each fixed \( i \), the maps \( m_{i,j} \) form a map of complexes from the tensor product of \( (\text{Sym}_i \ddot{G})^* \otimes \wedge^{g+i} \ddot{F} \) with the Koszul complex to the Eagon-Northcott complex. For this it suffices to show that the maps \( \ddot{m}_{i,j} \) themselves yield commutative diagrams of vector spaces:

\[
\begin{align*}
(\text{Sym}_{i-j+1} \ddot{G})^* \otimes \wedge^{g+i-j+1} \ddot{F} & \xrightarrow{e} (\text{Sym}_{i-j} \ddot{G})^* \otimes \wedge^{g+i-j} \ddot{F} \otimes W \\
& \xrightarrow{d} (\text{Sym}_i \ddot{G})^* \otimes \wedge^{g+i} \ddot{F} \otimes \wedge^j W^* \otimes W
\end{align*}
\]

in which the maps \( e \) come from the differentials of the Eagon-Northcott complex and the maps \( d \) come from the differentials in the Koszul complex. Given the definitions above this is a straightforward computation. \( \blacksquare \)

### 3 Syzygy ideals of determinantal ideals

Recall that a map

\[ \phi : F \rightarrow G \]

of free modules over a polynomial ring \( S = \text{Sym}_4(W) \) which is represented by a matrix of linear forms corresponds to a map of vector spaces \( \ddot{F} \longrightarrow \ddot{G} \otimes W \), or equivalently to a pairing

\[ \phi' : \ddot{F} \otimes \ddot{G}^* \longrightarrow W. \]

We call \( \phi \) or \( \phi' \) \( 1 \)-generic if for every “pure” element \( 0 \neq a \otimes b \in \ddot{F} \otimes \ddot{G}^* \) we have \( \phi'(a \otimes b) \neq 0 \). The maximal minors of a \( 1 \)-generic map necessarily generate an ideal of the expected codimension (see Eisenbud [1988]).
**Theorem 3.1** Suppose that $S$ is a polynomial ring over a field $k$, and that $\phi : F \rightarrow G$ is a map of free $S$-modules of ranks $f$ and $g$ respectively, $f \geq g$, represented by a matrix of linear forms. Suppose further that the determinantal ideal $I := I_g(\phi)$ has codimension $f-g+1$. If the syzygy ideal of each nonzero element of $\text{Tor}_{f-g}(I, k)$ is $I$, then the map $\phi$ is 1-generic; if $g = 2$, then the converse also holds.

**Proof** By Theorem 2.1, we must show that the map $\phi$ is 1-generic if the pairing

$$
\mu_{f-g, f-g} : (\text{Sym}_{f-g}(\tilde{G}))^* \otimes \wedge^{f-g} W^* \longrightarrow \wedge^g \tilde{F}
$$

takes $\tau \otimes \wedge^{f-g} W^*$ onto $\wedge^g \tilde{F}$ for every $\tau \in (\text{Sym}_{f-g}(\tilde{G}))^*$. Choosing a generator for $\wedge^g \tilde{F}$, we may identify $\wedge^g \tilde{F}$ with $\wedge^{f-g} \tilde{F}^*$. Rearranging the tensor factors, $\mu_{f-g, f-g}$ yields a map $\wedge^{f-g} \tilde{F} \otimes (\text{Sym}_{f-g}(\tilde{G}))^* \longrightarrow \wedge^{f-g} W$ which is easily seen to be a composite

$$
\wedge^{f-g} \tilde{F} \otimes (\text{Sym}_{f-g}(\tilde{G}))^* \longrightarrow \wedge^{f-g} (\tilde{F} \otimes \tilde{G}^*) \longrightarrow \wedge^{f-g} W,
$$

where the first map is the part of the Cauchy decomposition dual to the one described in Section 3 and the second is the $(f-g)^{th}$ exterior power of the pairing $\phi : \tilde{F} \otimes \tilde{G}^* \longrightarrow W$ corresponding to $\phi$. Simplifying the notation somewhat and generalizing the situation by dropping the dependence of the power $f-g$ on the ranks of $F$ and $G$, the desired result follows from:

**Theorem 3.2** Let $\alpha : A \otimes B^* \longrightarrow C$ be a pairing of vector spaces over a field $k$. Let

$$
\alpha_n : \wedge^n A \otimes (\text{Sym}_n B)^* \longrightarrow \wedge^n C
$$

be the composition of the dual Cauchy map $\wedge^n A \otimes (\text{Sym}_n B)^* \longrightarrow \wedge^n (A \otimes B^*)$ with the $n^{th}$ exterior power of $\alpha$.

a) If $\alpha_n$ is 1-generic for some $n$ between 1 and the rank of $A$, then $\alpha = \alpha_1$ is 1-generic.

b) If $\text{rank } B \leq 2$, then the converse also holds.

**Proof** Identify $(\text{Sym}_n B)^*$ with the $n^{th}$ divided power of $B^*$, and write the $n^{th}$ divided power of an element $b \in B^*$ as $b^{(n)}$. One checks first that the restriction of $\alpha_n$ to the subspace $\wedge^n A \otimes b^{(n)}$ is the $n^{th}$ exterior power of the map which is the restriction of $\alpha$ to $A \otimes b$.

To prove part a), suppose that $\alpha$ is not 1-generic, so that some nonzero vector $a \otimes b$ goes to 0. Then $\alpha_n(a \wedge a_2 \wedge \cdots \wedge a_n \otimes b^{(n)}) = 0$ for any elements $a_2, \ldots, a_n$ and thus part a) follows.

Conversely, suppose that $\alpha$ is 1-generic, and rank $B = 2$. We may harmlessly replace $C$ by the image of $\alpha$. From the classification theory of 1-generic $2 \times m$ matrices (see Eisenbud-Harris [1987], Harris [1992], or directly
from the Kronecker-Weierstrass theory of matrix pencils, see Gantmacher [1959,1986]) it follows that $A$ may be identified as a direct sum of the form

$$A = \text{Sym}_{d_1}(B^*) \oplus \ldots \oplus \text{Sym}_{d_m}(B^*),$$

and $C$ may be identified as the direct sum

$$C = \text{Sym}_{d_{1+1}}(B^*) \oplus \ldots \oplus \text{Sym}_{d_{m+1}}(B^*),$$

in such a way that the map $\alpha$ is the direct sum of the multiplication maps

$$\text{Sym}_{d_i}(B^*) \otimes B^* \longrightarrow \text{Sym}_{d_{i+1}}(B^*).$$

To prove part b), we use a degeneration argument: Suppose that the nonzero pure vector $a \otimes b$ goes to zero under $\alpha$, We may without loss of generality suppose that the ground field $k$ is infinite, so with a general choice of generators $s, t$ of $B^*$ we may suppose that $b = s^{[d]} + \text{(lower order)}$ involves a pure divided power of $s$. Taking the limit of a suitable one-parameter subgroup of $GL(2,k)$, we may actually suppose that $b = s^{[d]}$. The remark at the beginning of the proof finishes the argument.

We conjecture that the “converse” part of Theorem 3.1 holds without restriction on $g$, and consequently that Theorem 3.2 is true without restriction on the rank of $B$. We can prove this for the syzygies coming from “pure” vectors in $\text{Tor}_f^g(I, k) = (\text{Sym}_{f-g}G)^*$. One way to formulate the missing step is this:

**Conjecture 3.3** Let $\psi$ be a $p \times q$-matrix of linear forms over a polynomial ring $T$ and let $n$ denote the minimal dimension of the span of any generalized row of $\psi$. If $k \leq q - n + 1$ and $k \leq p$, then $\binom{n+k-1}{k}$ is the minimal dimension of the span of any generalized row of $\wedge^k \psi$.

It is not hard to reduce Conjecture 3.3 to the following special case: If every generalized row of $\psi$ consists of linear forms that span the space of linear forms of $T$, then every generalized row of $\wedge^k \psi$, with $k$ as above, spans the $k^{th}$ power of the maximal ideal of $T$. The case of two variables is easy, following the ideas above; we do not have a proof in the case of three variables.

The proof of Theorem 3.1 just given is similar to something implicit in Green’s proof [1996] of the linear syzygy conjecture. The syzygy variety is the image of an exterior minors map in the sense used in that paper.

## 4 Syzygetic Castelnuovo Lemma

Recall that a finite subscheme $\Gamma \subset \mathbf{P}^r$ is said to be in **linearly general position** if for every hyperplane $H$ of $\mathbf{P}^r$ the scheme $H \cap \Gamma$ has length $\leq r$. 
In Eisenbud-Harris [1992] it was shown that if \( k \) is an algebraically closed field and the length of \( \Gamma \) is \( \leq r + 3 \), then \( \Gamma \) lies on a rational normal curve of degree \( r \) if and only if \( \Gamma \) is in linearly general position. (The case when \( \Gamma \) is reduced is the original “Castelnuovo Lemma”.) See Eisenbud-Popescu [1998] for a version that works without algebraic closure. For large numbers of points one needs an additional condition, which can be expressed in terms of syzygies:

**Theorem 4.1** *(Syzygetic Castelnuovo Lemma for schemes).* A finite subscheme \( \Gamma \subset \mathbb{P}^n_k \) of length \( \geq r + 3 \) over an algebraically closed field \( k \) lies on a smooth rational normal curve if and only if

1. \( \Gamma \) contains a subscheme of length \( r + 3 \) in linearly general position;
2. \( \text{Tor}_{r-2}(I \Gamma, k)_r \neq 0 \).

Theorem 4.1 was proved for reduced \( \Gamma \) by Green [1984], and Yanagawa remarked in [1994] that the scheme theoretic version follows, given the results of Eisenbud-Harris [1992]. The full statement also follows from the results of Eibauer [1994]. Here we give a direct proof, using a technique related to that of Green’s recent proof of the “Linear Syzygy Conjecture” [1996]. In Green’s terms, we identify a certain syzygy ideal with an ideal of exterior minors.

In fact if we assume that \( \Gamma \) contains a (locally) Gorenstein subscheme of degree \( r + 3 \) in linearly general position, then one can replace the reference to Eisenbud-Harris in the proof below with a reference to Eisenbud-Popescu [1998], and omit the hypothesis of algebraic closure. One should compare the result with the much easier result describing when a projective scheme has \( \text{Tor}_{r-1}(I \Gamma, k)_r \neq 0 \). See also Cavaliere, Rossi, Valla [1995] for some variations.

**Proposition 4.2** Let \( I \) be any ideal in the polynomial ring in \( r + 1 \) variables \( S = k[x_0, \ldots, x_r] \). We have \( \text{Tor}_{r-1}(I, k)_r \neq 0 \) if and only if \( I \) contains the ideal of \( 2 \times 2 \) minors of a rank \( 2 \) matrix of the form

\[
\begin{pmatrix}
x_0 & x_1 & \cdots & x_r \\
l_0 & l_1 & \cdots & l_r
\end{pmatrix}
\]

where the \( l_i \) are linear forms, and the row of \( l_i \) is not a scalar multiple of the row of \( x_i \). In this case, if \( k \) is algebraically closed, the scheme associated to \( I \) lies on the union of nontrivial linear subspaces \( L_1 \) and \( L_2 \) with \( \text{codim} L_1 + \text{codim} L_2 = r + 1 \). In particular, the forms in \( I \) do not vanish on any set of \( r + 2 \) points in linearly general position.

**Proof** The first statement of Proposition 4.2 follows directly from the computation of Koszul cohomology. To deduce the second statement, note
that the matrix is too large to be 1-generic given that $k$ is algebraically
closed and the number of variables is only $r + 1$ (Eisenbud [1988]). Thus
after row and column operations and a linear change of variables we may
suppose that for some $i < r$ the forms $l_0, \ldots , l_i$ are linearly independent
while $l_{i+1} = \ldots = l_r = 0$. It follows that the scheme associated to $I$ is
contained in the union of the $(r - i - 1)$-plane $l_0 = \ldots = l_i = 0$ and the
$i$-plane $x_{i+1} = \ldots = x_r = 0$.

For the proof of Theorem 4.1 we use the following remark:

**Lemma 4.3** Let $M \subset N$ be graded modules over a graded ring $S =
k \oplus W \oplus \ldots$. If the lowest degree of a generator of $N$ is $d$, then for any
$i \geq 0$ the natural map $\text{Tor}_i(M, k)_{d+i} \longrightarrow \text{Tor}_i(N, k)_{d+i}$ is an inclusion,
and if $r \in \text{Tor}_i(M, k)_{d+i}$ is any element, then the syzygy submodule of $r$ in
$M$ is equal to the syzygy submodule of $r$ in $N$.

**Proof of Theorem 4.1** The ideal of a rational normal curve $C$ in $\mathbb{P}^r$ is in
suitable coordinates generated by the ideal of $2 \times 2$ minors of the 1-generic
matrix
\[
\begin{pmatrix}
x_0 & x_1 & \ldots & x_{r-1} \\
x_1 & x_2 & \ldots & x_r
\end{pmatrix}
\]
(see for example Harris [1992]). Its resolution is given by the Eagon-North-
cott complex; in particular, the $i^\text{th}$ syzygies of $I_C$ have degree $i + 2$. If $\Gamma$
is a subscheme of $C$ of length at least $r + 1$, then $I_{\Gamma}$ contains no linear
forms. By Lemma 4.3 the natural morphisms $\text{Tor}_i(I_C, k) \longrightarrow \text{Tor}_i(I_{\Gamma}, k)$
are monomorphisms, and thus in particular $\text{Tor}_{r-2}(I_{\Gamma}, k)_r \neq 0$.

For the converse, suppose $\Lambda$ is a subscheme of length $r + 3$ of $\Gamma$, and
that $\Lambda$ is in linearly general position. By Eisenbud-Harris [1992] there is a
rational normal curve $C$ containing $\Lambda$. It is easy to compute syzygies for
any subscheme of a rational normal curve (see for example Eisenbud [1995
Appendix A2.21].) In the case of a subscheme of length $r + 3$, it follows
that $\text{Tor}_{r-2}(I_{\Lambda}, k)_r = \text{Tor}_{r-2}(I_C, k)_r$. Since $I_{\Lambda}$ does not contain any linear
forms, we have by Lemma 4.3 that $\text{Tor}_{r-2}(I_{\Lambda}, k)_r \subset \text{Tor}_{r-2}(I_C, k)_r$. If $\tau \in \text{Tor}_{r-2}(I_{\Gamma}, k)_r$, then we may regard $\tau$ as a syzygy of $I_{\Lambda}$ and then of
$I_C$. As a syzygy of $I_C$, its syzygy ideal is all of $I_C$, by Theorem 3.1. By the
Lemma 4.3, its syzygy ideal as a syzygy of $I_{\Lambda}$ or of $I_{\Gamma}$ is the same. Thus
$I_{\Gamma}$ contains $I_C$.

We close this note with an application to another well-known result:

**Proposition 4.4** Let $\Gamma \subset \mathbb{P}^r$ be a finite subscheme of degree $r + 2$. If $\Gamma$
is in linearly general position, then the homogeneous coordinate ring $S_{\Gamma}$ is
Gorenstein.

**Proof** Write $S = k[x_0, \ldots , x_r]$ for the homogeneous coordinate ring of $\mathbb{P}^r$.
The ideal of $\Gamma$ is easily be seen to be 2-regular, and since it contains no linear
forms the generators of the canonical module $\omega_\Gamma = \text{Ext}^r(S_\Gamma, S(-r - 1))$
can only be in degrees $-1$ and $0$. If $\omega_\Gamma$ had a generator in degree $0$, then
we would have $\text{Tor}_{r-1}(I, k)_r \neq 0$; applying Proposition 4.2 we derive a
contradiction from the fact that $\Gamma$ is in linearly general position. Thus $\omega_\Gamma$
has all its generators in degree $-1$. Computing Hilbert functions we see that
there is just one generator in this degree, so $S_\Gamma$ is Gorenstein as required.

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