



Cohomology on Toric Varieties and Local Cohomology with Monomial Supports

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In this note we describe aspects of the cohomology of coherent sheaves on a complete toric variety X over a field k and, more generally, the local cohomology, with supports in a monomial ideal, of a finitely generated module over a polynomial ring S . This leads to an efficient way of computing such cohomology, for which we give explicit algorithms.

The problem is finiteness. The i th local cohomology of an S -module P with supports in an ideal B is the limit

$$H_B^i(P) = \varinjlim_{\ell} \text{Ext}^i(S/B_{\ell}, P),$$

where B_{ℓ} is any sequence of ideals that is cofinal with the powers of B . We will be interested in the case where S is a polynomial ring, P is a finitely generated module, and B is a monomial ideal. The module on the left of this equality is almost never finitely generated (even when $P = S$), whereas the module $\text{Ext}^i(S/B_{\ell}, P)$ on the right is finitely generated, so that the limit is really necessary.

We can sometimes restore finiteness by considering the homogeneous components of $H_B^i(P)$ for a suitable grading. For example, in the case where B is a monomial ideal and $P = S$ the modules on both sides are \mathbf{Z}^n graded, and for each $p \in \mathbf{Z}^n$ the component $H_B^i(S)_p$ is a finite-dimensional vector space. From this and the Hilbert syzygy theorem, one easily sees that the corresponding finiteness holds for any finitely generated \mathbf{Z}^n -graded module P .

To obtain a computation that works for more than \mathbf{Z}^n -graded modules, we work with coarser gradings; that is we consider a grading in an abelian group \mathbf{D} that is a homomorphic image of \mathbf{Z}^n . Of course if the grading is too coarse, we will lose finiteness again. In this paper we give a criterion on the grading for such finiteness to hold. When it holds we study the convergence of the limit above, and show how to compute, for each $\delta \in \mathbf{D}$, an explicit ideal $B^{[\ell]}$ such that the natural map

$$\text{Ext}^i(S/B^{[\ell]}, P)_{\delta} \longrightarrow H_B^i(P)_{\delta}$$

is an isomorphism. The computation involves the solution of a linear programming problem that depends on the grading \mathbf{D} and on the syzygies of P .

The results in this paper were motivated by the problem of computing the cohomology of coherent sheaves on a complete toric variety, and such varieties provide the most

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interesting cases for which our finiteness condition is satisfied. The connection is via Cox’s (1995) notion of the homogeneous coordinate ring of a toric variety, described in Section 2. The homogeneous coordinate ring is a polynomial ring $S = k[x_1, \dots, x_n]$ equipped with a grading in an abelian group \mathbf{D} (the group of invariant divisor classes on X) and an *irrelevant ideal* $B = B_X$ generated by square-free monomials, defined from the fan associated to X . The data B and \mathbf{D} satisfy our finiteness condition.

Following ideas of Cox explained in Section 2, we may represent any coherent sheaf on a toric variety X by giving a \mathbf{D} -graded module on a polynomial ring S , and the cohomology of the sheaf is given by a formula similar to that for local cohomology above. Thus, we obtain an explicit computation of sheaf cohomology in this case.

In the first section of this paper we treat various general results on local cohomology with monomial supports. We show that the local cohomology $H_B^i(S)_p$, for $p \in \mathbf{Z}^n$, depends only on which coordinates of p are negative. We analyse the condition that for a coarser grading $\mathbf{Z}^n \rightarrow \mathbf{D}$ the homogeneous components $H_B^i(S)_\delta$ are finite dimensional for all $\delta \in \mathbf{D}$ —this condition is satisfied for example when B and \mathbf{D} come from the homogeneous coordinate ring of a complete toric variety. In the second section we translate these results to the case of toric varieties, and also give a new topological characterization of the $p \in \mathbf{Z}^n$ for which $H_B^i(S)_p \neq 0$.

In Section 3, assuming finiteness, we determine ℓ such that the natural map $\text{Ext}^i(S/B^{[\ell]}, S)_\delta \rightarrow H_B^i(S)_\delta$ is an isomorphism, and similarly for sheaf cohomology. In Section 4 we use syzygies to extend this to all modules P .

Section 5 is devoted to algorithms made from these results.

In Section 6 we present some basic problems, partially solved in this paper in the monomial case: when are the maps $\text{Ext}^i(S/I, S) \rightarrow H_I^i(S)$ monomorphisms? In general, given an ideal B , we do not even know that there are ideals I with the same radical as B such that this map is a monomorphism, and we display a method proposed by Huneke that gives a non-existence criterion.

Using the computation of sheaf cohomology via local cohomology and the results above, Mustață (unpublished data) has proved a cohomology vanishing result on toric varieties including one that strengthens a version of the well-known vanishing theorem of Kawamata and Viehweg in this case, and is valid in all characteristics. The main point is again the fact that $H_B^i(S)_p$, for $p \in \mathbf{Z}^n$ depends only on which coordinates of p are negative. His result is:

THEOREM 0.1. *Let X be an arbitrary toric variety and D an invariant Weil divisor. If there is $E = \sum_{j=1}^d a_j D_j$, with $a_j \in \mathbf{Q}$, $0 \leq a_j \leq 1$ and D_j prime invariant Weil divisors such that for some integers $m \geq 1$ and $i \geq 0$, $m(D + E)$ is integral and Cartier and*

$$H^i(\mathcal{O}_X(D + m(D + E))) = 0,$$

then $H^i(\mathcal{O}_X(D)) = 0$. In particular, if X is complete and there is E as before such that $D + E$ is \mathbf{Q} -ample, then $H^i(\mathcal{O}_X(D)) = 0$, for all $i \geq 1$.

An important step toward the theorems of this paper is a result of Mustață (2000), motivated by this project. It shows that the local cohomology of $M = S$ with monomial supports can be computed as a union, not just as a limit, of suitable Ext modules. For the reader’s convenience we state it below. We write $B^{[\ell]}$ for the ideal generated by the ℓ th powers of monomials in B .

THEOREM 0.2. *Let $B \subset S = k[x_1, \dots, x_n]$ be a square-free monomial ideal. For each integer $j \geq 0$ the natural map*

$$\text{Ext}^j(S/B^{[\ell]}, S) \longrightarrow H_B^j(S)$$

is an injection, and its image is the submodule of $H_B^j(S)$ consisting of all elements of degree $\geq (-\ell, -\ell, \dots, -\ell)$.

It would be nice to have such results for more general ideals B ; see Section 6 for some remarks on this problem.

Henceforward in this paper k denotes an arbitrary field, and B denotes a reduced monomial ideal in a polynomial ring $S = k[x_1, \dots, x_n]$.

1. The Support of $H_B^i(S)$

In this section we establish our basic results for the local cohomology of the ring with supports in a monomial ideal B . Since the local cohomology depends only on the radical of B , we may assume that B is generated by square-free monomials. In the next section we explain the parallel theory for the cohomology of sheaves on a toric variety, with some refinements possible only in that case.

As B is a monomial ideal, the local cohomology $H_B^i(S)$ is naturally \mathbf{Z}^n -graded. Our main goal is to describe the set of indices $p \in \mathbf{Z}^n$ for which $H_B^i(S)_p \neq 0$. We first show that this condition depends only on which components of p are negative. Set $\text{neg}(p) = \{i \in \{1, \dots, n\} \mid p_i < 0\}$.

THEOREM 1.1. *If $p, q \in \mathbf{Z}^n$ satisfy $\text{neg}(p) = \text{neg}(q)$, then there is a canonical isomorphism*

$$H_B^i(S)_p \cong H_B^i(S)_q.$$

PROOF. We use the computation of local cohomology as Čech cohomology. Given a sequence of elements $\mathbf{f} = \{f_1, \dots, f_r\}$ of a ring R , and an R -module P , the Čech complex $C(\mathbf{f}, P)$ is the complex with

$$C^i(\mathbf{f}, P) = \bigoplus_{j_1 < \dots < j_i} P_{f_{j_1} f_{j_2} \dots f_{j_i}},$$

and differential

$$\partial^i : C^i(\mathbf{f}, P) \longrightarrow C^{i+1}(\mathbf{f}, P)$$

which is the alternating sum of the localization maps. The i th cohomology group of this complex is the i th local cohomology of P with supports in the ideal generated by the f_i ; see, for example, Brodmann and Sharp (1998).

Let (m_1, \dots, m_r) be monomial generators for the ideal B , and write \mathbf{m} for the sequence m_1, \dots, m_r . Given $p \in \mathbf{Z}^n$, let $C(\mathbf{f}, S)_p$ denote the complex of vector spaces that is the degree p part of the Čech complex $C(\mathbf{m}, S)$. Let m_J be the least common multiple of the monomials $m_j, j \in J$. It is easy to check that

$$C(\mathbf{m}, S)_p = \bigoplus_{\{J \mid \text{neg}(p) \subset \text{supp}(m_J)\}} k,$$

with differential mapping the J th component to the J' th component equal to zero unless

J' has the form $J' = J \cup j$, while in this case it is $(-1)^e$ where e is the position of j in the set J' .

Thus the complex $C(\mathbf{m}, S)_p$ depends only on $\text{neg}(p)$. The homology of this complex is $H_B^i(S)_p$. \square

In view of Theorem 1.1 it is useful to define

$$L_I = \{p \in \mathbf{Z}^n \mid \text{neg}(p) = I\}.$$

If we write $\text{Supp}(M)$ for the set of \mathbf{Z}^n -degrees in which a \mathbf{Z}^n -graded module M is nonzero, then part of Theorem 1.1 asserts that $\text{Supp}(H_B^i(S))$ is a union of certain L_I . We set

$$\Sigma_i = \{I \subset \{1, \dots, n\} \mid H_B^i(S)_p \neq 0 \text{ for } p \in L_I\}.$$

It will also be useful to define $C_I \subset \mathbf{Z}^n$ to be the orthant

$$C_I = \left\{ p \in \mathbf{Z}^n \mid \begin{cases} p_i \leq 0 & \text{if } j \in I \\ p_i \geq 0 & \text{if } j \notin I \end{cases} \right\}$$

and

$$(p_I)_j = \begin{cases} -1 & \text{if } j \in I \\ 0 & \text{if } j \notin I. \end{cases}$$

Note that $L_I = p_I + C_I \subset C_I$.

If Δ is a simplicial complex on $\{1, \dots, n\}$, the Alexander dual Δ^* is defined to be the simplicial complex

$$\Delta^* := \{\{1, \dots, n\} \setminus f \mid f \notin \Delta\},$$

and the Alexander dual of a square-free monomial ideal B is the corresponding notion for Stanley–Reisner ideals: B^* is the monomial ideal generated by the square-free monomials in

$$(x_1^2, \dots, x_n^2) : B.$$

Part (c) of the following corollary seems to be the most efficient way to compute the Σ_i .

COROLLARY 1.2. *The following are equivalent.*

- (a) $I \in \Sigma_i$
- (b) $\text{Ext}^i(S/B, S)_{p_I} \neq 0$
- (c) $\text{Tor}_j^S(S/B^*, k)_{-p_I} \neq 0$, where $j = \#I - i + 1$.

PROOF. By Mustaă’s theorem (Theorem 0.2) we have $\text{Ext}^i(S/B, S)_{p_I} = H_B^i(S)_{p_I}$ and Theorem 1.1 gives the equivalence of (a) and (b). The equivalence of (b) and (c) is Corollary 3.1 in Mustaă (2000). \square

Now we introduce a grading coarser than the \mathbf{Z}^n -grading. Let

$$0 \rightarrow M \xrightarrow{\rho} \mathbf{Z}^n \xrightarrow{\phi} \mathbf{D} \rightarrow 0$$

be an exact sequence of abelian groups. Any \mathbf{Z}^n -graded module N can be regarded as a \mathbf{D} -graded module by setting $N_\delta = \bigoplus_{p \in \phi^{-1}\delta} N_p$ for each $\delta \in \mathbf{D}$. We are interested in the

finiteness of $H_B^i(S)_\delta$ for all $\delta \in \mathbf{D}$. From the Čech complex (or from Theorem 0.2) we see that $H_B^i(S)_p$ is a finite-dimensional vector space for all $p \in \mathbf{Z}^n$. Thus we have the following.

COROLLARY 1.3. *$(p + M) \cap L_I$ is finite for every $I \in \Sigma_i$ and every $p \in \mathbf{Z}^n$, if and only if $H_B^i(S)_\delta$ is a finite-dimensional vector space for every $\delta \in \mathbf{D}$.*

The finiteness of the components $H_B^i(S)_\delta$ translates into a condition on the subsets I in the Σ_i and the subgroup $M \subset \mathbf{Z}^n$ defining the coarse grading:

PROPOSITION 1.4. *With notation as above, let I' be the complement of I . The following are equivalent.*

- (a) *For every $p \in \mathbf{Z}^n$, the set $(p + M) \cap L_I$ is finite.*
- (b) *$M \cap C_I = 0$.*
- (b') *$M \cap C_{I'} = 0$.*
- (c) *For any $p, q, r \in \mathbf{Z}^n$ such that $q \neq r \in (p + M) \cap C_I$, there are indices i, j such that $|q_i| > |r_i|$ and $|q_j| < |r_j|$.*
- (d) *For every $p \in \mathbf{Z}^n$ the set $(p + M) \cap C_I$ is finite.*
- (d') *For some $p \in \mathbf{Z}^n$, the set $(p + M) \cap C_I$ is finite and nonempty.*

PROOF. (a) \Rightarrow (b): Suppose on the contrary that $0 \neq p \in C_I \cap M$. Choose $q \in L_I$. Then $q + sp \in (q + M) \cap L_I$ for every positive integer s , showing that $(q + M) \cap L_I$ is not finite. (b) \equiv (b'): $M \cap C_I = -(M \cap C_{I'})$. (b) \Rightarrow (c): If all the coordinates of q had absolute value \geq the corresponding coordinates of r then $q - r \in M \cap C_I$, and similarly for the other inequality. (c) \Rightarrow (d): Suppose contrary to (c) that $(p + M) \cap C_I$ is infinite. Any set of elements in an orthant contains a finite set of minimal elements with respect to the partial order by absolute values of the coordinates, and every element is \geq to one of these in the partial order. (Proof: Moving to the first quadrant we may think of the elements as monomials in the polynomial ring. By the Hilbert basis theorem, a finite subset generates the same ideal as the whole set.) Thus two elements of $(p + M) \cap C_I$ are comparable, contradicting (b). (d) \Rightarrow (d'): It is enough to pick $p \in C_I$, so that $(p + M) \cap C_I$ is nonempty. (d') \Rightarrow (b): If $q \in (p + M) \cap C_I$ and $r \in M \cap C_I$ then $q + r \in (p + M) \cap C_I$. Thus if $(p + M) \cap C_I$ is finite then $M \cap C_I$ is finite. As the latter is a cone, it must be 0. (d) \Rightarrow (a): This is obvious since $L_I \subset C_I$. \square

Of particular importance to us is the image of the cones L_I (with $I \in \Sigma_i$) in \mathbf{D} . The following result is what ultimately allows us to understand the convergence of $\text{Ext}^i(S/B^{[\ell]}, S)_\delta$ to $H_B^i(S)_\delta$ for $\delta \in \mathbf{D}$.

COROLLARY 1.5. *If the condition in Corollary 1.3 is satisfied, then for $I \in \Sigma_i$ the projection $\phi(C_I)$ is a pointed cone in the sense that if $x, -x \in \phi(C_I)$ then $x = 0$. Further, p_I maps to a nontorsion element of D .*

PROOF. It is easy to see that $\phi(C_I)$ contains both a nonzero vector and its negative if and only if the kernel of ϕ meets an open face of the cone C_I without containing it. By part (b) in Proposition 1.4 the kernel M meets only the zero face. \square

2. The Case of Toric Varieties

Let X be a toric variety which is always assumed to be nondegenerate, meaning that the corresponding fan is not contained in a hyperplane. In this section we will introduce the technique that we will use to handle sheaves on X , and we describe the set of torus invariant divisors D for which $H^i(\mathcal{O}_X(D))$ is nonzero.

We begin by reviewing Cox’s homogeneous coordinate ring, and the basic computation of cohomology that it allows. Let Δ be the fan in \mathbf{Z}^d corresponding to X (see, for example, Fulton, 1993, for notation and terminology), and suppose that x_1, \dots, x_n are the edges (one-dimensional cones) of Δ . The torus-invariant divisor classes correspond to the elements of the cokernel \mathbf{D} of the map represented by a matrix whose rows are the coordinates of the first integral points of the n edges of Δ , so that we have an exact sequence

$$M := \mathbf{Z}^d \xrightarrow{\rho} \mathbf{Z}^n \xrightarrow{\phi} \mathbf{D}.$$

As before, the map ϕ defines a grading by \mathbf{D} on the polynomial ring $S := k[x_1, \dots, x_n]$

Cox (1995) defines the *homogeneous coordinate ring* of X to be the polynomial ring S together with the \mathbf{D} -grading and the *irrelevant ideal*

$$B = \left(\left\{ \prod_{x_i \notin \sigma} x_i \mid \sigma \in \Delta \right\} \right).$$

Given a \mathbf{D} -graded S -module P , Cox constructs a quasicoherent sheaf \tilde{P} on X by localizing just as in the case of projective space. Coherent sheaves come from finitely generated modules:

THEOREM 2.1. *Every coherent \mathcal{O}_X module may be written as \tilde{P} , for a finitely generated \mathbf{D} -graded S -module P .*

A proof is given by Cox (1995) in the simplicial case and in Mustařa (unpublished data) in general.

For any \mathbf{D} -graded S -module P and any $\delta \in \mathbf{D}$ we may define $P(\delta)$ to be the graded module with components $P(\delta)_\epsilon = P_{\delta+\epsilon}$ and we set

$$H_*^i(\tilde{P}) = \bigoplus_{\delta \in \mathbf{D}} H^i(\widetilde{P(\delta)}).$$

We have $H^0(\mathcal{O}_X(\delta)) = S_\delta$ for each $\delta \in \mathbf{D}$. In general we write

$$H_*^i(\mathcal{O}_X) = \bigoplus_{\delta \in \mathbf{D}} H^i(\mathcal{O}_X(\delta))$$

so that $H_*^0(\mathcal{O}_X) = S$. In fact each $H_*^i(\mathcal{O}_X)$ is a \mathbf{Z}^n -graded S -module. We can compute $H_*^i(\tilde{P})$ in terms of local cohomology or limits of Ext modules. The results and proofs are easy generalizations of results in Grothendieck (1967).

We will also consider the shifted Čech complex $C_{[1]}(\mathbf{f}, P)$, where $C_{[1]}^i(\mathbf{f}, P) = C^{i+1}(\mathbf{f}, P)$ for $i \geq 0$ and $C_{[1]}^{-1}(\mathbf{f}, P) = 0$. If P and the f_i are all \mathbf{D} -graded then each map in these complexes is graded of degree zero, and so the cohomology modules will also be \mathbf{D} -graded.

THEOREM 2.2. *Let P be a \mathbf{D} -graded S -module. Let \tilde{P} be the corresponding quasi-coherent sheaf on X as above. If the irrelevant ideal $B \subset S$ is generated by the monomials m_i then*

$$H_*^i(\tilde{P}) \cong H^i(C_{[1]}(\mathbf{m}, P)),$$

as graded S -modules.

The proof is immediate from the definition of $\tilde{P}(\delta)$ and the computation of cohomology as Čech cohomology.

PROPOSITION 2.3. *With P and B as in the theorem,*

(a) *For $i \geq 1$, there is an isomorphism of graded S -modules*

$$H_*^i(\tilde{P}) \cong H_B^{i+1}(P).$$

(b) *There is an exact sequence of graded S -modules*

$$0 \longrightarrow H_B^0(P) \longrightarrow P \longrightarrow H_*^0(\tilde{P}) \longrightarrow H_B^1(P) \longrightarrow 0.$$

PROOF. Both assertions follow from the fact that

$$H_B^i(P) \cong H^i(C(\mathbf{m}, P)). \square$$

This concludes our review of basic machinery.

Owing to the similarity of the computation of sheaf cohomology and local cohomology in this setting, we can translate the results of the last section directly. For example, translating Theorem 1.1 we have:

THEOREM 2.4. *The module $H_*^i(\mathcal{O}_X)$ is \mathbf{Z}^n -graded. If $p, q \in \mathbf{Z}^n$ satisfy $\text{neg}(p) = \text{neg}(q)$, then there is a canonical isomorphism*

$$H_*^i(\mathcal{O}_X)_p \cong H_*^i(\mathcal{O}_X)_q.$$

Thus we can define for each $i \geq 0$,

$$\Sigma_i^{[1]} = \{I \subset \{1, \dots, n\} \mid H_*^i(\mathcal{O}_X)_{p_I} \neq 0\},$$

and we have:

COROLLARY 2.5.

$$\text{Supp } H_*^i(\mathcal{O}_X) = \bigcup_{I \in \Sigma_i^{[1]}} L_I.$$

For a complete toric variety $H^i(\mathcal{O}_X(D))$ is finite dimensional for every invariant divisor D of X , so the conditions of Proposition 1.4 hold. In the general toric setting we can provide a more explicit finiteness condition to complement those given in Proposition 1.4. We will call a one-dimensional cone of the fan Δ an *edge*.

PROPOSITION 2.6. *With notation as in Proposition 1.4, if B and \mathbf{D} come from a toric variety X with fan Δ as above, then $H^i(\mathcal{O}_X(D))$ is finite dimensional for every invariant divisor D of X if and only if for every $I \in \Sigma_i^{[1]}$, and every $p \in \mathbf{Z}^n$, $(p + M) \cap L_I$ is finite. Conditions (a)–(d) of Proposition 1.4 are also equivalent in this case to:*

(e) *There is no hyperplane $H \subset N_{\mathbf{R}} = \mathbf{R}^d$ such that every edge of Δ indexed by an element of I lies in or on one side of H and every edge indexed by an element of I' lies in or on the other side of H .*

PROOF. The first statement is simply the application of Proposition 1.4 to our case. We continue the notation of Proposition 1.4 and prove (b) \equiv (e): If $0 \neq p \in M \cap C_I$ is regarded as a linear functional on N_R , then its hyperplane of zeros satisfies the given condition. Conversely, a hyperplane H that satisfies the condition separates the two cones spanned by the edges indexed by I and I' . As these cones are both spanned by integral vectors, we may take H to be integral, and an integral functional p vanishing on H is in $M \cap C_I$. \square

In the toric case, the actual value of $H_*^i(\mathcal{O}_X)_p$ can be computed from a topological formula. Note that Fulton (1993) gives a slightly different topological description in the special case where the divisor in X corresponding to p is Cartier.

If Δ is the fan of X , then we write $|\Delta|$ for the union of the cones in Δ . For a subset $I \subset \{1, \dots, n\}$, we define Y_I to be the union of those cones in Δ having all the edges in the complement of I (if $I = \{1, \dots, n\}$, we take $Y_I = \{0\}$). We use the notation $H_{Y_I}^i(|\Delta|) = H^i(|\Delta|, |\Delta| \setminus Y_I)$. Since $|\Delta|$ is contractible, we have $H_{Y_I}^i(|\Delta|) = H^{i-1}(|\Delta| \setminus Y_I)$ (reduced cohomology with coefficients in k). We have:

THEOREM 2.7. *With the above notation, if $p \in \mathbf{Z}^n$ and $I = \text{neg}(p)$, then*

$$H_*^i(\mathcal{O}_X)_p \cong H_{Y_I}^i(|\Delta|).$$

PROOF. If $i = 0$, $H_*^0(\mathcal{O}_X) = S$, so that $H_*^0(\mathcal{O}_X)_p \cong k$ if $I = \emptyset$ and 0 otherwise. Since the same thing holds trivially for $H_{Y_I}^0(|\Delta|)$, we can conclude this case.

Suppose now that $i \geq 1$. In this case,

$$H_*^i(\mathcal{O}_X)_p \cong H_B^{i+1}(S)_p.$$

We will use the topological description of $H_B^{i+1}(S)_p$ for any square-free monomial ideal B , from Mustața (2000, Theorem 2.1). It states that if m_1, \dots, m_r are the minimal monomial generators of B , $I = \text{neg}(p)$ and T_I is the simplicial complex on $\{1, \dots, r\}$ given by

$$T_I = \{J \subset \{1, \dots, r\} \mid X_i \nmid \text{lcm}(m_j; j \in J) \text{ for some } i \in I\},$$

then

$$H_B^{i+1}(S)_p \cong H^{i-1}(T_I, k). \tag{*}$$

When $I = \emptyset$, T_I is defined as the void complex. Note that in (*) and everywhere below, the cohomology groups are reduced and with coefficients in k .

In our situation the minimal generators of B correspond exactly to the maximal cones in Δ . Therefore, if $\sigma_1, \dots, \sigma_r$ are the maximal cones in Δ , then T_I is the simplicial complex on $\{1, \dots, r\}$ given by: $J \in T_I$ if and only if there is $i \in I$ such that $X_i \nmid X_{\hat{\sigma}_j}$, for every $j \in J$. Equivalently, J is in T_I if and only if the intersection of the cones in J has some edge in I .

If $I = \emptyset$, then the conclusion follows trivially from (*), since the void complex has trivial reduced cohomology.

Therefore we may suppose that $I \neq \emptyset$. We have already seen that $H_{Y_I}^i(|\Delta|) \cong H^{i-1}(|\Delta| \setminus Y_I)$.

Let us consider the following cover of $|\Delta| \setminus Y_I$:

$$\{\sigma \setminus Y_I \mid \sigma \in \Delta \text{ maximal cone}\}.$$

$(\sigma_{i_1} \setminus Y_I) \cap \dots \cap (\sigma_{i_k} \setminus Y_I) \neq \emptyset$ if and only if $\sigma_{i_1} \cap \dots \cap \sigma_{i_k} \not\subseteq Y_I$. This means precisely that $\sigma_{i_1} \cap \dots \cap \sigma_{i_k}$ has an edge in I . Therefore T_I is the nerve of the above cover.

On the other hand, if $(\sigma_{i_1} \setminus Y_I) \cap \dots \cap (\sigma_{i_k} \setminus Y_I) \neq \emptyset$, then this set is equal to $\sigma \setminus (\sigma \cap Y_I)$, where $\sigma = \sigma_{i_1} \cap \dots \cap \sigma_{i_k} \in \Delta$. However, a sharp cone minus some of its proper faces is contractible. Therefore by Godement (1958, Theorem 5.2.4), we obtain

$$H^{i-1}(|\Delta| \setminus Y_I) \cong H^{i-1}(T_I)$$

and finally, using (*),

$$H_*^i(\mathcal{O}_X)_p \cong H_{Y_I}^i(|\Delta|). \square$$

EXAMPLE 2.8. Let X be a (not necessarily complete) toric surface. In this case Theorem 2.7 gives a complete description of $\Sigma_i^{[1]}$.

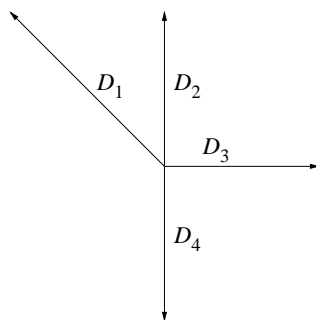
For $i = 0$, $H_{Y_I}^0(|\Delta|) = 0$, unless $Y_I = |\Delta|$ i.e. $I = \emptyset$, in which case $H_{Y_I}^0(|\Delta|) \cong k$. Therefore $\Sigma_0^{[1]} = \{\emptyset\}$.

For $i = 2$, $H_{Y_I}^2(|\Delta|) \cong H^1(|\Delta| \setminus Y_I)$, which is zero, unless X is complete and $I = \{1, \dots, n\}$, in which case $H_{Y_I}^2(|\Delta|) \cong k$. Hence $\Sigma_2^{[1]} = \{\{1, \dots, n\}\}$ if X is complete and it is empty otherwise.

The interesting case is $i = 1$, when $H_{Y_I}^1(|\Delta|) \cong H^0(|\Delta| \setminus Y_I)$, so that $\dim_k H_{Y_I}^1(|\Delta|)$ is the number of connected components of $|\Delta| \setminus Y_I$ minus one, unless $Y_I = |\Delta|$, in which case it is zero.

If X is complete, then $I \notin \Sigma_1^{[1]}$ if and only if the one-dimensional cones in I form a sequence D_1, \dots, D_n such that D_i and D_{i+1} are adjacent for $1 \leq i < n$.

In particular, we see that it is possible to have some I satisfying the equivalent conditions in Proposition 2.6, but such that $I \notin \Sigma_i^{[1]}$, for all i . Take, for example the complete toric surface associated with the following fan:



Then $\{1, 3, 4\}$ has the above property: it satisfies condition (e) of Proposition 2.6.

3. Bounding $H_B^i(S)_\delta$ and $H^i \mathcal{O}_X(D)$

In this section we assume that B and D are such that the local cohomology $H_B^i(S)_\delta$ is finite dimensional for all $\delta \in \mathbf{D}$. The motivating examples are those when B and \mathbf{D} come from the homogeneous coordinate ring of a complete toric variety X . We will derive an effective means of calculating any $H_B^i(S)_\delta$ or $H^i \mathcal{O}_X(D)$ in terms of an Ext. The bounds we need come from Corollary 1.3.

For simplicity we give the results in this section only for the local cohomology case. For the corresponding results on sheaf cohomology one merely replaces $H_B^i(S)_\delta$ by $H^i(\mathcal{O}_X(D))$ (where D is a torus-invariant divisor representing δ) and Σ_i by Σ_i^1 . At the end of this section we illustrate this translation by working out the example of the two-dimensional rational normal scrolls (Hirzebruch surfaces).

We adopt p_I, C_I, Σ_i and Σ_i^1 of the previous sections.

PROPOSITION 3.1. *If $I \in \Sigma_i$, and $p \in \mathbf{Z}^n$ then*

$$(p + M) \cap (p_I + C_I)$$

is (contained in) a bounded polyhedron. If we define $f_{I,j}(\delta)$ to be the maximum of zero and the negative of the minimum value of the j th coordinates of the points in $(p + M) \cap (p_I + C_I)$, where p is any representative of $\delta \in \mathbf{D}$, then

$$\text{Ext}^i(S/B^{[\ell]}, S)_\delta = H_B^i(S)_\delta$$

if and only if $\ell \geq \max\{f_{I,j}(\delta) \mid I \in \Sigma_i, j \in I\}$.

PROOF. Let $Q = (p+M) \cap (p_I+C_I)$. The first statement follows since $|Q| \leq \dim H_B^i(S)_\delta < \infty$, so Q consists of the integral points of a convex bounded polyhedron. The last statement follows by Theorems 2.4 and 0.2. \square

One may find the bound $f_{I,j}(\delta)$ in various ways. For example, one may solve a linear programming problem in $M \otimes \mathbf{R}$. An equivalent definition of $f_{I,j}$ is the following: $f_{I,j}(\delta)$ is the least non-negative integer ℓ such that

$$\delta \notin \phi(-\ell e_j + L_I).$$

If \mathbf{D} has no torsion, then one may find $f_{I,j}(\delta)$ exactly by using the facet equations of the cone $\phi(C_I)$. If \mathbf{D} has torsion, this method still gives a bound, although possibly not the exact value.

PROPOSITION 3.2. *If $C_I \cap M = 0$ then the absolute value of each coordinate of a point of $(p + M) \cap (p_I + C_I)$ is bounded above by*

$$\frac{d^2 \max_i \{|p_i|\} Q_1 Q_{d-1}}{q_d},$$

where q_d is the minimum of the nonzero absolute values of the $d \times d$ minors of ρ ($q_d \geq 1$, as ρ is a matrix of integers) and Q_j is the maximum of the absolute values of the $j \times j$ minors of ρ .

PROOF. Since $p_I + C_I \subset C_I$ it suffices to bound the points of $(p + M) \cap C_I$. Since this is a bounded polyhedron, the coordinates of its points, as vectors of M , are bounded by the coordinates of its vertices. Each vertex $u \in M$ is defined by a system of d linear equations in the d coordinates of u having the form $\rho' u = -p$, where ρ' is a $d \times d$ submatrix of ρ with nonvanishing determinant. Thus the i th coordinate u_i has the form

$$\frac{\sum_{j=1}^d -p_j a_j}{\det \rho'}$$

where the a_j are $(d - 1) \times (d - 1)$ minors of ρ' , and thus has absolute value bounded

by $(d \max_i \{|p_i|\} Q_{d-1})/q_d$. Applying ρ , we obtain a vector in \mathbf{Z}^n whose coordinates have absolute value at most dQ_1 times this, whence the given bound. \square

COROLLARY 3.3. *Suppose $p \in \mathbf{Z}^n$ is a representative for $\delta \in \mathbf{D}$. If*

$$\ell = \frac{d^2 \max_i \{|p_i|\} Q_1 Q_{d-1}}{q_d}$$

is the bound in Proposition 3.2 then

$$\mathbf{H}_B^i(S)_\delta = \text{Ext}_S^i(S/B^{[\ell]}, S)_\delta.$$

PROOF. Combine Propositions 3.1 and 3.2. \square

COROLLARY 3.4. *Fix an integer $i \geq 0$, and an element $\delta \in \mathbf{D}$. The inclusion*

$$\text{Ext}^i(S/B^{[\ell]}, S)_\delta \longrightarrow \mathbf{H}_B^i(S)_\delta$$

is an isomorphism if and only if

$$\ell \geq f_{I,j}(\delta),$$

for all $I \subset \Sigma_i$ and all $j \in I$ where $f_{I,j}(\delta)$ is the bound defined in Proposition 3.1.

The same result may be looked upon from a different point of view:

COROLLARY 3.5. *Fix an integer $i \geq 0$, and an integer $\ell \geq 0$. The set of $\delta \in \mathbf{D}$ such that*

$$\text{Ext}^i(S/B^{[\ell]}, S)_\delta \longrightarrow \mathbf{H}_B^i(S)_\delta$$

is an isomorphism is the complement (in \mathbf{D}) of the union of the translated pointed cones

$$\phi(-\ell e_j) + \phi(L_I),$$

for all $I \in \Sigma_i$ and all $j \in I$.

EXAMPLE 3.6. (SURFACE SCROLLS) Choose a non-negative integer $e \geq 0$. Let

$$0 \longrightarrow \mathbf{Z}^2 \xrightarrow{\rho} \mathbf{Z}^4 \xrightarrow{\phi} \mathbf{D} = \mathbf{Z}^2 \longrightarrow 0.$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ -1 & e \\ 0 & -1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & e \end{pmatrix}$$

The matrix ρ defines a fan $\Delta \subset \mathbf{R}^2$, with $X = X(\Delta)$ a rational normal surface scroll. The ring $S = k[x_1, x_2, x_3, x_4]$ has the \mathbf{Z}^2 -grading $\deg(x_1) = \deg(x_3) = (0, 1)$, $\deg x_2 = (1, 0)$, and $\deg x_4 = (1, e)$. The irrelevant ideal $B = (x_1, x_3) \cap (x_2, x_4)$. The nonzero Ext modules of S/B (shown with their fine gradings) are

$$\text{Ext}^2(S/B, S) = S(-1, 0, -1, 0)/(x_1, x_3) \oplus S(0, -1, 0, -1)/(x_2, x_4),$$

and

$$\text{Ext}^3(S/B, S) = S(-1, -1, -1, -1)/(x_1, x_2, x_3, x_4).$$

Therefore the nonempty sets Σ_i are

$$\Sigma_1^{[1]} = \Sigma_2 = \{\{1, 3\}, \{2, 4\}\},$$

and

$$\Sigma_2^{[1]} = \Sigma_3 = \{\{1, 2, 3, 4\}\}.$$

To compute the cohomology module $H^i(\mathcal{O}_X(a, b))$, one may use $p = (b, a, 0, 0) \in \mathbf{Z}^4$, and the bound in Corollary 3.3 becomes

$$\ell = 4e^2 \max\{|a|, |b|\}.$$

This is in general not best possible. For $i = 0$, one may always take $\ell = 0$. For $i = 2$, by solving for the minimum of any coordinate of any element of $(p + M) \cap (p_I + C_I)$, where $I = \{1, 2, 3, 4\}$, one finds that

$$\ell = \max(-a - 1, -b - e - 1, 0)$$

is the best possible bound.

For $i = 1$, we have two sets I to consider. For $I = \{1, 3\}$, the set of (a, b) for which $(p + M) \cap (P_I + C_I)$ is nonempty is the set

$$\{(a, b) \mid a \geq 0, b \leq ae - 2\}.$$

The corresponding bound is the minimum of coordinates 1,3 of any point in $(p + M) \cap (P_I + C_I)$. This works out to a bound of

$$\ell = \max(-b + ae - 1, 0),$$

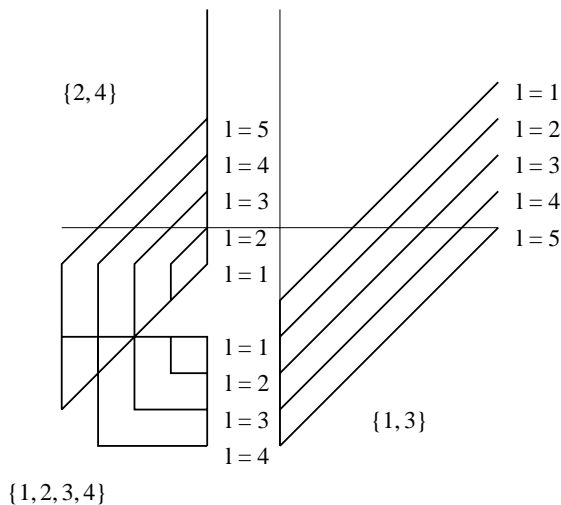
for any (a, b) in the above set. The second set, $I = \{2, 4\}$, gives rise to a region

$$\{(a, b) \mid a \leq -2, b \geq ae + e\}.$$

The corresponding bound for points in this region is

$$\ell = \max(-a - 1, b/e - a, 0).$$

For the case $e = 1$. The illustration below shows the cones $\phi(L_I)$, for $I = 1, 3, 2, 4$, and $1, 2, 3, 4$. In addition, for each of these three translated cones, the lines $\ell = r$ are shown, for $r = 2, 3, 4, 5$. For $\ell = 1$, the only points in these three cones such that $\text{Ext}^i(S/B^{[\ell]}, S)_\delta = H_B^i(S)_\delta$ are the vertices of the cones.



4. Bounds for $H_B^i(P)_\delta$ and $H^i(\tilde{P}(\delta))$

Once we know bounds for computing the local cohomology of the ring S , we can deduce the bounds for the case of an arbitrary finitely generated graded module. This is done in the proposition below using a spectral sequence argument which has the same flavor as the arguments in Smith (2000).

PROPOSITION 4.1. *Suppose that we have functions $f_i : \mathbf{D} \rightarrow \mathbf{N}^*$ such that:*

$$\text{Ext}_S^i(S/B^{[k]}, S)_\delta \longrightarrow H_B^i(S)_\delta$$

is an isomorphism for every $\delta \in \mathbf{D}(X)$, $i \geq 0$ and $k \geq f_i(\delta)$. Let P be a finitely generated \mathbf{D} -graded S module. Let F_\bullet be a free resolution of P with

$$F_i = \bigoplus_{\alpha \in \mathbf{D}(X)} S(-\alpha)^{\beta_{i,\alpha}}.$$

Then

$$\text{Ext}_S^i(S/B^{[k]}, P)_\delta \longrightarrow H_B^i(P)_\delta$$

is an isomorphism if for every $j \geq 0$ and every α such that $\beta_{j,\alpha} \neq 0$ we have

$$k \geq \max\{f_{i+j-1}(\delta - \alpha), f_{i+j}(\delta - \alpha)\}$$

if $j \geq 1$ and $k \geq f_i(\delta - \alpha)$ if $j = 0$.

PROOF. Let E^\bullet be the complex given by $E^{-i} = F_i$, $i \geq 0$. We will use the two spectral sequences of hypercohomology for the functors $\text{Hom}(S/B^{[k]}, -)$ and $H_B^0(-)$ and the complex E^\bullet (see Weibel, 1994, for details about hypercohomology).

For $\text{Hom}(S/B^{[k]}, -)$ we obtain the spectral sequences:

$$\begin{aligned} {}'E_2^{p,q} &= H^p(\text{Ext}_S^q(S/B^{[k]}, E^\bullet)) \Rightarrow \mathbf{Ext}^{p+q}(S/B^{[k]}, E^\bullet) \\ {}''E_2^{p,q} &= \text{Ext}_S^p(S/B^{[k]}, H^q(E^\bullet)) \Rightarrow \mathbf{Ext}_S^{p+q}(S/B^{[k]}, E^\bullet). \end{aligned}$$

Since $H^q(E^\bullet) = 0$ for $q \neq 0$ and $H^0(E^\bullet) = P$, the second spectral sequence collapses and the first one becomes:

$${}'E_2^{p,q} = H^p(\text{Ext}_S^q(S/B^{[k]}, E^\bullet)) \Rightarrow \text{Ext}_S^{p+q}(S/B^{[k]}, P).$$

Applying the same argument for $H_B^0(-)$, we obtain the spectral sequence:

$${}'\overline{E}_2^{p,q} = H^p(H_B^q(E^\bullet)) \Rightarrow H_B^{p+q}(P).$$

Moreover, the natural map $\text{Hom}(S/B^{[k]}, -) \rightarrow H_B^0(-)$ induces a natural map of spectral sequences. Therefore, in order for the map

$$\text{Ext}_S^i(S/B^{[k]}, P)_\delta \longrightarrow H_B^i(P)_\delta$$

to be an isomorphism, it is enough to have

$$u^{p,q} : H^p(\text{Ext}_S^q(S/B^{[k]}, E^\bullet))_\delta \longrightarrow H^p(H_B^q(E^\bullet))_\delta$$

isomorphism for every p and q with $p \leq 0$ and $p + q = i$.

In the diagram below:

$$\begin{array}{ccccc}
 \text{Ext}_S^q(S/B^{[k]}, E^{p-1})_\delta & \longrightarrow & \text{Ext}_S^q(S/B^{[k]}, E^p)_\delta & \longrightarrow & \text{Ext}_S^q(S/B^{[k]}, E^{p+1})_\delta \\
 \downarrow v^{p-1,q} & & \downarrow v^{p,q} & & \downarrow v^{p+1,q} \\
 H_B^q(E^{p-1})_\delta & \longrightarrow & H_B^q(E^p)_\delta & \longrightarrow & H_B^q(E^{p+q})_\delta
 \end{array}$$

the vertical maps are injective by Theorem 0.1. Therefore, $u^{p,q}$ is an isomorphism if both $v^{p,q}$ and $v^{p-1,q}$ are isomorphisms.

Since $E^p = \bigoplus_\alpha S(-\alpha)^{\beta-p,\alpha}$, using the properties of the functions f_i we obtain the conclusion of the proposition. \square

Translating this to sheaf cohomology is immediate:

COROLLARY 4.2. *Suppose that we have functions $f_i : \mathbf{D} \rightarrow \mathbf{N}^*$ such that:*

$$\text{Ext}_S^i(B^{[\ell]}, S)_\delta \longrightarrow H^i(\mathcal{O}_X(\delta))$$

is an isomorphism for every $\delta \in \mathbf{D}$, $i \geq 0$ and $\ell \geq f_i(\delta)$. Let P be a finitely generated graded S module. Let F_\bullet be a minimal free resolution of P with

$$F_i = \bigoplus_{\alpha \in \mathbf{D}} S(-\alpha)^{\beta_i, \alpha}.$$

Then

$$\text{Ext}_S^i(B^{[\ell]}, P)_\delta \longrightarrow H_*^i(\tilde{P})_\delta$$

is an isomorphism if for every $j \geq 0$ and every α such that $\beta_{j,\alpha} \neq 0$ we have

$$k \geq \max\{f_{i+j-1}(\delta - \alpha), f_{i+j}(\delta - \alpha)\}$$

if $j \geq 1$ and $k \geq f_i(\delta - \alpha)$ if $j = 0$.

5. Algorithms and Examples

In this section we give explicit algorithms corresponding to the theorems in the previous section, and then give some examples. These algorithms have been implemented in the *Macaulay 2* system (Grayson and Stillman, 1993).

Given the ideal B and the grading $\phi : \mathbf{Z}^n \rightarrow \mathbf{D}$, we eventually compute $\Sigma_i = \Sigma_i(B)$, the \mathbf{Z}^n -support of $\text{Ext}_S^i(B, S)$, as well as the bounding hyperplanes for the cones $\phi(C_I)$, for each $I \in \Sigma_i$.

The following algorithm is essentially Corollary 3.1 in Mustařă (2000).

ALGORITHM 5.1. ($\Sigma_i(B)$, all i)

input: A square-free monomial ideal $B \subset S$.

output: The \mathbf{Z}^n -support of the module $\text{Ext}_S^i(B, S)$, for each i .

begin

set $B^* :=$ Alexander dual of B :

$B^* :=$ the monomial ideal generated by the square-free monomials
in $(x_1^2, \dots, x_n^2) : B$

set $\Sigma_i := \emptyset$, for all i

for $i := 0$ **to** n **do**

```

set  $T := \text{Tor}_i(S/B^*, k)$ 
for each  $\mathbf{Z}^n$  degree  $p$  of  $T$  do
  set  $I := \{j \in 1..n \mid p_j \neq 0\}$ 
   $\Sigma_{(|p|-i+1)} := \Sigma_{(|p|-i+1)} \cup \{I\}$ 
return the sets  $\Sigma_i$ , for  $i = 1, \dots, n$ 
end.

```

The following routine corresponds to 3.4. The ideal B and grading ϕ are implicit parameters.

ALGORITHM 5.2. (BOUND (i, δ, S))

input: An integer $i \in \mathbf{Z}$, and a coarse degree $\delta \in \mathbf{D}$.

output: The least integer ℓ such that $\text{Ext}_S^i(S/B^{[\ell]}, S)$ in degree δ is equal to $H_B^i(S)_\delta$.

begin

set $\ell := 0$

for each I **in** $\Sigma_i(B)$ **do**

for each j **in** I **do**

 Solve the linear programming problem:

$m :=$ the negative of the minimum value of the j th coordinate in
 $(p + M) \cap L_I$

 where $p \in \mathbf{Z}^n$ is a representative of δ .

if $m > \ell$ **then set** $\ell := m$.

return ℓ

end.

In view of the remarks after Proposition 3.1, we may compute the bound m in the above algorithm by instead computing the facet equations of $\phi(C_I)$, and then using them to find the minimum $m \geq 0$ such that $\delta \notin \phi(p_I - me_j + C_I)$.

ALGORITHM 5.3. (BOUND (i, δ, F))

input: An integer $i \in \mathbf{Z}$, and a coarse degree $\delta \in \mathbf{D}$, and a graded free S -module

$$F = \bigoplus_{\alpha \in \mathbf{D}} S(-\alpha)^{\beta_\alpha}$$

output: The least integer ℓ such that $\text{Ext}_S^i(S/B^{[\ell]}, F)$ in degree δ is equal to $H_B^i(F)_\delta$.

begin

return the maximum of the numbers

$\{\text{bound}(i, \delta - \alpha, S) \mid \beta_\alpha \neq 0\}$

end.

ALGORITHM 5.4. (BOUND (i, δ, P))

input: An integer $i \in \mathbf{Z}$, a coarse degree $\delta \in \mathbf{D}$, and a graded S -module P .

output: An integer ℓ such that $\text{Ext}_S^i(S/B^{[\ell]}, P)$ in degree δ is equal to $H_B^i(P)_\delta$.

begin

 Compute a minimal graded free resolution F_\bullet of P .

set $\ell := \text{bound}(i, \delta, F_0)$

for each $j \geq 1$ **do**

```

set  $\ell := \max(\ell, \text{bound}(i + j, \delta, F_j))$ 
set  $\ell := \max(\ell, \text{bound}(i + j - 1, \delta, F_j))$ 
return  $\ell$ 
end.

```

For a specific module P , this bound is not always best possible. Once we have this bound ℓ , we compute $\text{Ext}^i(S/B^{[\ell]}, P)$, using standard methods. The degree δ part of this module is easily extracted using a Gröbner basis of this module, or the dimension may be found by computing its (multi-graded) Hilbert function.

EXAMPLE 5.5. Let Y be $\mathbf{P}(3, 3, 3, 1, 1, 1)$ and X its desingularization. Then the homogeneous coordinate ring of X is $S = k[X_1, \dots, X_6, T]$, $\mathbf{D}(X) = \mathbf{Z} \times \mathbf{Z}$ and $\text{deg}(X_i) = (3, 1)$ for $1 \leq i \leq 3$, $\text{deg}(X_i) = (1, 0)$ for $4 \leq i \leq 6$ and $\text{deg}(T) = (0, 1)$.

Let Δ_Y be the fan defining Y . Then the maximal cones of Δ are $\sigma_1, \dots, \sigma_6$, where σ_i is generated by $\bar{e}_1, \dots, \bar{e}_i, \dots, \bar{e}_6$ ($\bar{e}_1 + \dots + \bar{e}_6 = 0$ and the lattice N is generated by $1/3\bar{e}_1, 1/3\bar{e}_2, 1/3\bar{e}_3, \bar{e}_4, \bar{e}_5$ and \bar{e}_6). Let $\bar{f} = 1/3(\bar{e}_4 + \bar{e}_5 + \bar{e}_6)$ and for each i , $1 \leq i \leq 3$ we consider $\sigma_i = \sigma_{i4} \cup \sigma_{i5} \cup \sigma_{i6}$, where σ_{ij} is obtained by replacing \bar{e}_j with \bar{f} in σ_i . Then the maximal cones of the fan Δ defining X are σ_{ij} for $1 \leq i \leq 3$, $4 \leq j \leq 6$ and σ_i for $4 \leq i \leq 6$.

Note that in Δ any two edges are contained in a maximal cone, while the only three edges that do not belong to a maximal cone are $\{\bar{e}_4, \bar{e}_5, \bar{e}_6\}$. Using this and our topological description of the support we deduce that $\Sigma_i = \emptyset$ if $i \neq 3, 4$ or 6 , while

$$\begin{aligned} \Sigma_3 &= \{\{\bar{e}_4, \bar{e}_5, \bar{e}_6\}\}, \\ \Sigma_4 &= \{\{\bar{e}_1, \bar{e}_2, \bar{e}_3, \bar{f}\}\}, \\ \Sigma_6 &= \{\{\bar{e}_1, \dots, \bar{e}_6, \bar{f}\}\}. \end{aligned}$$

We obtain the functions $f_i : \mathbf{D}(X) \rightarrow \mathbf{N}^*$ that satisfy the property in Corollary 4.2: $f_i \equiv 1$ if $i \neq 3, 4$ or 6 , and

$$\begin{aligned} f_3(\delta_1, \delta_2) &= \begin{cases} 1, & \text{if } \delta_2 < \max(0, 1/3\delta_1 + 1); \\ 3\delta_2 - \delta_1 - 2, & \text{if } \delta_2 \geq \max(0, 1/3\delta_1 + 1), \end{cases} \\ f_4(\delta_1, \delta_2) &= \begin{cases} 1, & \text{if } \delta_2 + 1 > \min(-3, 1/3\delta_1); \\ -\delta_2 - 3, & \text{if } \delta_2 + 1 \leq \min(-3, 1/3\delta_1), \end{cases} \\ f_6(\delta_1, \delta_2) &= \begin{cases} 1, & \text{if } \delta_1 > -12 \text{ or } \delta_2 > -4; \\ \max(-\delta_2 - 3, -\delta_1 - 11), & \text{if } \delta_1 \leq -12 \text{ and } \delta_2 \leq -4. \end{cases} \end{aligned}$$

6. A Problem on Local Cohomology

Let S be a polynomial ring and let $I \subset S$ be an ideal. The natural maps $\text{Ext}_S^j(S/I^d, S) \rightarrow H_I^j(S)$ are rarely injections (they are almost never surjections). It is thus interesting and, from the point of view of computation, useful to ask when some analogue of Mustařa's Theorem 0.2 holds:

QUESTION 6.1. *For which ideals I does there exist a sequence of ideals*

$$I \supset I_1 \supset I_2 \supset \cdots \supset I_d \supset \cdots$$

such that each $\text{Ext}_S^j(S/I_d, S)$ injects into $H_I^j(S)$ and

$$H_I^j(S) = \bigcup \text{Ext}_S^j(S/I_d, S)?$$

An even more basic question is:

QUESTION 6.2. *For which ideals $I \subset S$ is the natural map $\text{Ext}^j(S/I, S) \rightarrow H_I^j(S)$ an inclusion?*

By Theorem 0.2, Frobenius powers of reduced monomial ideals do satisfy the condition of Question 6.2, and thus sequences of ideals as in Question 6.1 exist whenever I is monomial, but we do not know the full answer to Question 6.2 even for monomial ideals.

EXAMPLE 6.3. If we require the condition in Question 6.2 for all j , then I must be unmixed. (Proof: Take j equal to the codimension of the embedded prime. Both Ext and local cohomology localize, so we may begin by localizing and assume that the embedded prime is the maximal ideal. We then find that $\text{Ext}^j(S/I, S) \neq 0$, but $H_I^j(S) = 0$ as one sees by writing $H_I^j(S) = \lim \text{Ext}_S^j(S/I^{(\ell)}, S)$, where $I^{(\ell)}$ denotes the intersection of the ℓ th symbolic powers of the minimal primes of I .) However, even when S is a polynomial ring and I is a monomial ideal, this condition does not suffice. If $I \subset k[a, b, c, d]$ denotes the monomial ideal $(ab, acd, bd^2, cd^2) = (b, d) \cap (b, c) \cap (a, d^2)$ then one sees by computation that the map $\text{Ext}_S^3(S/I, S) \rightarrow \text{Ext}_S^3(S/I^2, S)$ has kernel k .

Craig Huneke pointed out to us one way that one might produce ideals I for which no sequence as in Question 6.1 can exist: let k be a field of characteristic $p > 0$, and let I be a prime ideal in S such that $H_I^j(S) = 0$ for $j > \text{codim}(I)$. Let J be any ideal with radical I . It follows that $\text{Ext}^*(S/J, S)$ injects in $H_I^*(S) = H_J^*(S)$ if and only if J is perfect—that is S/J is Cohen–Macaulay (in this case we say that I is set-theoretically Cohen–Macaulay). If such a J exists, then the ideal $J^{[p^n]}$ generated by the p^{nth} powers of elements of J is perfect too (apply the Frobenius to the whole free resolution of J , using the characterization that J is perfect if and only if it has a free resolution of length equal to $\text{codim}(J)$), so the ideals $J_d = J^{[p]}$ satisfy the condition of Question 6.1 above.

Suppose now that S/I is F -pure. We claim that if there exists a perfect ideal I with the same radical as J , then I would be perfect. To see this, suppose x, y is a system of parameters modulo I , and $r, s \in S$ are such that $rx + sy \in I$. It follows that

$$r^{p^m} x^{p^m} + s^{p^m} y^{p^m} \in J$$

for large m , and thus $s^{p^m} \in (x^{p^m} + J) \subset (x^{p^m} + I)$. From F -purity we obtain $s \in (x + I)$, so x, y is a regular sequence.

Thus, if I is an imperfect prime ideal such that S/I is F -pure and $H_I^j(S) = 0$ for all $j > \text{codim}(I)$ then no sequence as in Question 6.1 can exist for I . Unfortunately we do not know whether such a prime ideal exists. See Huneke and Lyubeznik (1990) for some results where at least the cohomology vanishing is proven.

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