A note on the Intersection of Veronese Surfaces

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0 Introduction

The main purpose of this note it to prove the following

Theorem 0.1 *Any two Veronese surfaces in* $\mathbb{P}^5$ *whose intersection is zero-dimensional meet in at most* 10 *points (counted with multiplicity).*

Our initial motivation for this note comes from our paper [EGHPO] where we study linear syzygies of homogeneous ideals generated by quadrics and their restriction to subvarieties of the ambient projective space with known (linear) minimal free resolution. A direct application of the techniques in [EGHPO, Section 3] shows that the homogeneous ideal of a zero-dimensional intersection of two Veronese surfaces in $\mathbb{P}^5$ is 5-regular (see also Lemma 1.1 below), which yields only an upper bound of 12 for its degree, cf. Section 1.

Section 2 analyzes Veronese surfaces on hyperquadrics. The observation that two Veronese surfaces on a smooth hyperquadric $Q \subset \mathbb{P}^5$ meeting in a zero-dimensional subscheme, must meet in a subscheme of length 10 or 6 is classical and goes back to Kummer [Ku] and Reye [Rey] (see also [Jes1] for historical comments): By regarding the smooth hyperquadric $Q \subset \mathbb{P}^5$ as the Plücker embedding of the Grassmannian of lines $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$, a Veronese surface on $Q \subset \mathbb{P}^5$ is, up to duality, the congruence of secant lines to a twisted cubic curve and thus has bidegree $(1, 3)$. More precisely, the congruence has one line passing through a generic point of $\mathbb{P}^3$ and 3 lines contained in a generic plane. Thus Schubert calculus yields that the possible intersection numbers of two Veronese surfaces on Q are either 10 or 6. See Proposition 2.1 below and the following remark, or the computation of the number of common chords of two space curves in [GH, page 297].

The case of two Veronese surfaces in $\mathbb{P}^5$ meeting in 10 simple points has also been investigated in relation with association (projective Gale transform) by Coble [Cob], Conner [Con] and others. In [Cob, Theorem 26] Coble claims that 10 points in $\mathbb{P}^5$ which are associated to the 10 nodes of a symmetroid in $\mathbb{P}^3$, the quartic surface defined by the determinant of a symmetric 4×4 matrix with linear entries in $\mathbb{P}^3$, are the (simple) intersection points of two Veronese surfaces in $\mathbb{P}^5$ (see Proposition 3.3 below). This is based on Reye's observation [Rey, page 78-79] that 4×4 symmetric matrices with linear entries in $\mathbb{P}^3$ are actually catalecticant with respect to suitable bases and on the analysis in [EiPo] of the Gale Transform of zero-dimensional determinantal schemes. Section 3 contains a modern account of these results.

In Section 4 we briefly discuss which intersection numbers ≤ 10 can actually occur for two Veronese surfaces in $\mathbb{P}^5$ and in which geometric situation this can happen. For instance, we show that two Veronese surfaces in $\mathbb{P}^5$ cannot intersect transversally in 9 points, however they may intersect in non-reduced zero-dimensional schemes of this degree.

With the exception of Section 3 all other results in this note are valid in arbitrary characteristic.

1 A reduction step

We shall make essential use of the following lemma whose proof is reminiscent of the linear syzygies techniques used in [EGHPO, Section 3].

Lemma 1.1 *If X_1 and X_2 are two Veronese surfaces in $\mathbb{P}^5$ meeting in a zero-dimensional scheme W, then the ideal sheaf $\mathcal{I}_{W,\mathbb{P}^2}$ of W regarded as a subscheme of $\mathbb{P}^2$ is 5-regular.*

Proof. The claim is equivalent to the vanishing $h^1(\mathcal{I}_{W,\mathbb{P}^2}(4)) = 0$. In order to see this we consider the minimal resolution of $\mathcal{I}_{X_1}$ in $\mathbb{P}^5$ and restrict it to X_2. This yields a complex abutting to $\mathcal{I}_W$. Since X_1 has property N_p for all p one immediately computes that $h^1(\mathcal{I}_{W,\mathbb{P}^2}(4)) = 0$. □

As immediate consequence we get a first bound for the number of points where two Veronese surfaces whose intersection is zero-dimensional can meet.

Proposition 1.2 *Two Veronese surfaces in $\mathbb{P}^5$ whose intersection is zero-dimensional meet in at most 12 points.*

Proof. Let X_1 and X_2 be two Veronese surfaces meeting in a zero-dimensional scheme W, and let $d = \text{length}(W)$. By Lemma 1.1, W regarded as a subscheme of $X_2 \cong \mathbb{P}^2$ imposes independent conditions on plane quartics, in particular if $d \leq 15$. On the other hand $X_2 \subset \mathbb{P}^5$ is cut out by quadrics scheme-theoretically, and thus the quartics in $H^0(\mathcal{I}_{W,\mathbb{P}^2}(4))$ must cut out $W \subset \mathbb{P}^2$ scheme-theoretically too – in particular there are at least two. If there were only two, then they would form a complete intersection, generating a saturated ideal, and thus $W \subset \mathbb{P}^2$ would be a complete intersection of two plane quartics, which cannot be 5-regular. It follows that $h^0(\mathcal{I}_{W,\mathbb{P}^2}(4)) \geq 3$, and thus that $d \leq 12$. □

Actually the above proof yields also the following estimate

Proposition 1.3 *Let X_1 and X_2 be two Veronese surfaces in $\mathbb{P}^5$ meeting in a zero-dimensional scheme W of length d with $10 \leq d \leq 12$. Then $X_1 \cup X_2$ lies on at least $d-9$ quadrics.*

Proof. Let $a = h^0(\mathcal{I}_{X_1 \cup X_2}(2))$. Then on one hand $h^0(\mathcal{I}_{W,\mathbb{P}^2}(4)) \geq h^0(\mathcal{I}_{X_2}(2)) - a = 6 - a$, on the other hand, by Lemma 1.1, we know that $h^0(\mathcal{I}_{W,\mathbb{P}^2}(4)) = 15 - d$. Combining the two proves the claim of the proposition. □

2 Veronese surfaces on hyperquadrics

In this section we analyze the intersection of two Veronese surfaces which meet in finitely many points, in the case where the two surfaces lie on a common hyperquadric, resp. a pencil of hyperquadrics. We begin with the classical case of congruences of lines:

Proposition 2.1 *Assume that X_1 and X_2 are Veronese surfaces which meet in finitely many points and assume moreover that there exists a smooth quadric hypersurface Q with $X_1 \cup X_2 \subset Q$. Then $X_1.X_2 = 6$ or 10.*

Proof. Since Q is smooth it is isomorphic to the Grassmannian $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ and it is well known that $H^4(\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3), \mathbb{Z}) = \mathbb{Z}\alpha + \mathbb{Z}\beta$ where α and β are 2-planes. It follows from the double point formula (see for instance [HS]) that every Veronese surface on $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3)$ has class $3\alpha + \beta$ or $\alpha + 3\beta$. Since $\alpha^2 = \beta^2 = 1$ and $\alpha\beta = 0$ it follows that $X_1 X_2 = 10$ or 6 depending on whether X_1 and X_2 belong to the same class or not. □

Remark As already mentioned in the introduction, a Veronese surface of class $3\alpha + \beta$ on the Grassmannian $\mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3) \cong Q \subset \mathbb{P}^5$ is the congruence of secant lines to a twisted cubic curve (where β is the cycle of lines passing through a point of $\mathbb{P}^3$ while α is the cycle of lines in a plane). Passing to the dual $\mathbb{P}^3$ exchanges α and β, so up to duality the same construction accounts also for Veronese surfaces of class $\alpha + 3\beta$. It is easy to see, for instance by using Kleiman's transversality theorem, that both cases described in Proposition 2.1 actually occur.

Proposition 2.2 *Assume that X_1 and X_2 are Veronese surfaces in $\mathbb{P}^5$ which meet in finitely many points and assume that there exists a rank 5 hyperquadric Q containing both X_1 and X_2. Then $X_1.X_2 = 8$.*

Proof. We first claim that X_1 and X_2 do not pass through the vertex P of the quadric cone Q. Otherwise projection from P would map the Veronese surface to a cubic scroll contained in a smooth hyperquadric $Q' \subset \mathbb{P}^4$. But this is impossible, since by the Lefschetz theorem every surface on Q' is a complete intersection and hence has even degree. Blowing up the point P we obtain a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{p} & Q \\ \downarrow{\scriptstyle \pi} & & \\ Q' & & \end{array}$$

where π gives $\tilde{Q}$ the structure of a $\mathbb{P}^1$-bundle over Q'. Since X_1 and X_2 do not go through the point P they are not blown up and we will, by abuse of notation, also denote their pre-images in $\tilde{Q}$ by X_1 and X_2. The Chow ring of $\tilde{Q}$ is generated by $H = p^*(H_{\mathbb{P}^5})$ and $H' = \pi^*(H_{\mathbb{P}^4})$. Clearly $H^4 = H^3H' = H^2(H')^2 = H(H')^3 = 2$ and $(H')^4 = 0$. Let E be the exceptional divisor of the map p. Then $E = \alpha H + \beta H'$ and from $EH^3 = 0$ and $E(H')^3 = 2$ one deduces $\alpha = 1$ and $\beta = -1$, i.e. $E = H - H'$. The surfaces X_i have class

$$X_i = \alpha_i H^2 + \beta_i HH' + \gamma_i (H')^2.$$

From $X_i H^2 = \deg X_i = 4$ one computes $\alpha_i + \beta_i + \gamma_i = 2$. Since X_i does not meet E we have $X_i EH' = 0$ and from this one deduces $\gamma_i = 0$ and hence

$$X_i = \alpha_i H^2 + (2 - \alpha_i) HH'.$$

But then $X_i^2 = 8$ and this proves the claim. □

We analyze next what happens when X_1 and X_2 lie on a pencil of hyperquadrics.

Proposition 2.3 *Assume that the Veronese surfaces X_1 and X_2 meet in finitely many points and assume that they are contained in a pencil of hyperquadrics $\{\lambda_1 Q_1 + \lambda_2 Q_2 = 0\}$. Then for a general hyperquadric Q in this pencil*

$$X_1 \cap X_2 \cap \operatorname{Sing} Q = \emptyset.$$

Proof. Assume that this is false. Since $X_1 \cap X_2$ is a finite set it follows that there exists some point $P \in X_1 \cap X_2$ which is singular for every hyperquadric Q in this pencil. From what we saw in the proof of Proposition 2.2 this also shows that the general quadric in this pencil has rank at most 4 (and at least 3 since both surfaces X_i are non-degenerate). Projecting from P maps X_1 and X_2 to rational cubic scrolls Y_1 and Y_2 in $\mathbb{P}^4$, respectively. These cubic scrolls are contained in a pencil of non-degenerate hyperquadrics $\{\lambda_1 Q'_1 + \lambda_2 Q'_2 = 0\}$ whose general member has rank 3 or 4. For degree reasons this implies $Y_1 = Y_2$. (Incidentally this also shows that X_1 and X_2 are contained in a net of hyperquadrics whose general element has rank 4.)

Let Y be the cone over $Y_1 = Y_2$ with vertex P. We obviously have $X_1, X_2 \subset Y$. We now blow up in P and obtain a diagram

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{p} & Y \subset \mathbb{P}^5 \\ \downarrow{\scriptstyle \pi} & & \\ Y_1 & & \end{array}$$

where $\tilde{Y}$ is a $\mathbb{P}^1$-bundle over Y_1. The Picard group of Y_1 is generated by two elements C_0 and F with $C_0^2 = -1, C_0 F = 1$ and $F^2 = 0$. Let $F_1 = \pi^* C_0$ and $F_2 = \pi^* F_1$. Then the Chow group on $\tilde{Y}$ is generated by $H = p^*(H_{\mathbb{P}^5}), F_1$ and F_2.

For geometric reasons $H^3 = 3, F_1^2 H = -1, F_1 F_2 H = F_1 H^2 = F_2 H^2 = 1$ and $F_2^2 H = F_1^2 F_2 = F_2^2 F_1 = 0$. Let E be the exceptional locus of p and let $\tilde{X}_i$ denote the strict transforms of the Veronese surfaces. Then π restricted to $\tilde{X}_i$ defines isomorphisms between $\tilde{X}_i$ and Y_1. Since X_1 and X_2 intersect in only finitely many points and since both are contained in the 3-dimensional cone Y it follows that $X_1 \cap X_2 = \{P\}$, and from this one concludes that $\tilde{X}_1 \cap \tilde{X}_2 = L$ where $L = E \cap F_1$ is a projective line, respectively $\tilde{X}_1 \tilde{X}_2 = aL$ for some $a \geq 1$. Next we want to determine the class of $\tilde{X}_i$ in $\tilde{Y}$. Since the $\tilde{X}_i$ are sections of the $\mathbb{P}^1$-bundle $\pi : \tilde{Y} \to Y$ we find $\tilde{X}_i = H + \beta_i F_1 + \gamma_i F_2; i = 1, 2$. Restricting this to E and using that H is trivial on E we immediately find that $\beta_i = 1$ and $\gamma_i = 0$, i.e. $\tilde{X}_i = H + F_1$. But then $\tilde{X}_1 \tilde{X}_2 = H^2 + 2HF_1 + F_1^2 \neq aL$ where the latter inequality can be seen e.g. by intersecting with H. This is a contradiction and the proposition is proved. $\square$

Proposition 2.4 *Let X_1, X_2 be two Veronese surfaces in $\mathbb{P}^5$ intersecting in a finite number of points. If X_1 and X_2 are contained in a pencil of hyperquadrics, then $X_1.X_2 \leq 10$.*

Proof. Let r be the rank of a general element of this pencil of hyperquadrics. If $r = 6$ or 5 then the assertion follows from Proposition 2.1, or from Proposition 2.2, respectively. On the other hand, since the surfaces X_i are non-degenerate we must have $r \geq 3$.

We shall first treat the case $r = 4$. According to Proposition 2.3 we can then choose a rank 4 hyperquadric Q with $X_1 \cap X_2 \cap \operatorname{Sing} Q = \emptyset$. Blowing up the singular line L of Q we obtain a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\quad p \quad} & Q \subset \mathbb{P}^5 \\ \downarrow{\scriptstyle \pi} & & \\ Q' := \mathbb{P}^1 \times \mathbb{P}^1 & & \end{array}$$

where π is the structure map of a $\mathbb{P}^2$-bundle. We denote the strict transforms of X_i by $\tilde{X}_i$, $i = 1, 2$. Since $X_1 \cap X_2 \cap \operatorname{Sing} Q = \emptyset$ we have $\tilde{X}_1.\tilde{X}_2 = X_1.X_2$. Let $H = p^*(H_{\mathbb{P}^5})$, let L_1, L_2 denote the rulings of $\mathbb{P}^1 \times \mathbb{P}^1$ and set $F_i = \pi^* L_i$, $i = 1, 2$. Then $H^4 = 2, F_1 H^3 = F_2 H^3 = F_1 F_2 H^2 = 1$ and $F_1^2 = F_2^2 = 0$. Let E be the exceptional locus of p. Its class must be of the form $E = H + \alpha_1 F_1 + \alpha_2 F_2$. From $E F_1 H^2 = H^2 = E F_2 H^2 = 0$ one deduces $\alpha_1 = \alpha_2 = -1$, i.e. $E = H - F_1 - F_2$.

Now let X be any Veronese surface on Q. We want to determine the possible classes of the strict transform $\tilde{X}$ of X in $\tilde{Q}$. Let $\tilde{X} = \alpha H^2 + \beta_1 F_1 H + \beta_2 F_2 H + \gamma F_1 F_2$. From $\tilde{X} H^2 = 4$ we obtain $2\alpha + \beta_1 + \beta_2 + \gamma = 4$. A priory the singular line L can either be disjoint from X, meet it transversally in one point, be a proper secant or a tangent of X. Projection from L shows that only the first and the third of these possibilities can occur.

Assume first that L and X are disjoint. Then $\pi_{|\tilde{X}} : \tilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ is a $2:1$ map which shows $\alpha = 2$. From $\tilde{X} E F_1 = \tilde{X} E F_2 = 0$ we conclude $\beta_1 = \beta_2 = 0$ and hence $\gamma = 0$, i.e. $\tilde{X} = 2H^2$.

Assume now that L is a proper secant of X. Blowing up Q along L then blows up X in 2 points and the corresponding exceptional curves are mapped to different rulings in $\mathbb{P}^1 \times \mathbb{P}^1$. The map $\pi_{|\tilde{X}} : \tilde{X} \to \mathbb{P}^1 \times \mathbb{P}^1$ is now birational and hence $\alpha = 1$. From what we have just said it follows that

$\tilde{X}EF_1 = \tilde{X}EF_2 = 1$ and hence this implies $\beta_1 = \beta_2 = 1$. But then $\gamma = 0$ and $\tilde{X} = H^2 + HF_1 + HF_2$.

Let $c_1 = 2H^2$ and $c_2 = H^2 + HF_1 + HF_2$. The claim of the proposition now follows for the rank 4 case since $c_1^2 = c_2^2 = c_1c_2 = 8$.

It remains to deal with the case in which the general hyperquadric Q in the pencil has rank 3. By Proposition 2.3 we can again assume that $X_1 \cap X_2 \cap \operatorname{Sing} Q = \emptyset$. Projection from the singular locus of Q gives a diagram

$$\begin{array}{ccc} \tilde{Q} & \xrightarrow{\;p\;} & Q \subset \mathbb{P}^5 \\ \downarrow{\scriptstyle \pi} & & \\ C \cong \mathbb{P}^1 & & \end{array}$$

where C is a conic section and $\pi : \tilde{Q} \to C$ is a $\mathbb{P}^3$-bundle. We denote $H = p^*(H_{\mathbb{P}^5})$ and $F = \pi^*(pt)$. Then $H^4 = 2, H^3F = 1$ and $F^2 = 0$. Let X be any Veronese surface on Q and denote its strict transform on $\tilde{Q}$ by $\tilde{X}$. We first note that $X \cap \operatorname{Sing} Q$ is a finite set. Otherwise $X \cap \operatorname{Sing} Q$ would have to be a conic section and projection from $\operatorname{Sing} Q$ would map X onto a plane, not to a conic. Finally let E be the exceptional locus of p. The class of E must be of the form $E = H + \gamma F$ and from $EH^3 = 0$ it follows that $\gamma = -1$, i.e. $E = H - F$. Now put $\tilde{X} = \alpha H^2 + \beta H.F$. From $\tilde{X}H^2 = 4$ one deduces that $2\alpha + \beta = 4$. Since $X \cap \operatorname{Sing} Q$ is finite one must have that $\tilde{X}EH = 0$ and hence $\beta = 0$. This shows that $\tilde{X} = 2H^2$ and the claim of the proposition follows since $X_1.X_2 = \tilde{X}_1.\tilde{X}_2 = 4H^4 = 8$. □

Remark The above proof shows that if the general element in the pencil of hyperquadrics containing X_1 and X_2 has rank 3 or 4, then $X_1.X_2 = 8$.

3 Catalecticant symmetroids and Veronese surfaces

In this section we prove Coble's claim [Cob, Theorem 26], mentioned in the introduction, that ten points in $\mathbb{P}^5$ which are the Gale transform of the nodes of a general quartic symmetroid in $\mathbb{P}^3$ are the simple intersection points of two Veronese surfaces.

A *quartic symmetroid* is the quartic surface in $\mathbb{P}^3$ defined by the determinant of a symmetric 4×4 matrix with linear entries in $\mathbb{P}^3$; for general choices (of the matrix) the symmetroid has only ordinary double points as singularities (nodes) and their number is 10, by Porteous' formula. These surfaces are sometimes called *Cayley symmetroids*, as Cayley initiated their study in [Cay] (cf. [Jes2], but see [Cos] for a modern account of Cayley's results and much more).

A symmetric matrix whose diagonals are constant is called a *catalecticant* matrix. Surprisingly enough, it turns out that a symmetric 4×4 matrix with linear entries in $\mathbb{P}^3$ can always be reduced to a catalecticant form (with respect to suitable bases). This fact goes back to Reye [Rey, page 78-79] and Conner [Con, page 39] and is (re)-proved below.

We will make use of the perfect pairing, called *apolarity*, between forms of degree n and homogeneous differential operators of order n induced by the action of $T = k[\partial_0, \dots, \partial_r]$ on $S = k[x_0, \dots, x_r]$ via differentiation:

$$\partial^\alpha(x^\beta) = \alpha!\binom{\beta}{\alpha}x^{\beta-\alpha},$$

if $\beta \geq \alpha$ and 0 otherwise, and where α and β are multi-indices, $\binom{\beta}{\alpha} = \prod\binom{\beta_i}{\alpha_i}$, and k is a field of characteristic zero.

Proposition 3.1 *The Hessian matrix of a web of quadrics in $\mathbb{P}^3$ is catalecticant (with respect to a suitable basis) if and only if the quadrics in the web annihilate the quadrics of a twisted cubic curve. (One says in this situation that the web is "orthic" to the twisted cubic curve.)*

Proof. Let $q : W^* \hookrightarrow \mathrm{Sym}_2 V$ be the web of quadrics on $\mathbb{P}^3 = \mathbb{P}(V)$. A twisted cubic $C \subset \check{\mathbb{P}}^3 = \mathbb{P}(V^*)$ is defined by its quadrics $H^0(\check{\mathbb{P}}^3, \mathcal{I}_C(2))$. In suitable coordinates, say $\partial_0, \dots, \partial_3$, these are the minors of the matrix

$$\begin{pmatrix} \partial_0 & \partial_1 & \partial_2 \\ \partial_1 & \partial_2 & \partial_3 \end{pmatrix}.$$

In terms of the dual coordinates, $x_0, \dots, x_3$ of $\mathbb{P}(V)$, the web q has the form

$$a_0x_0^2 + a_4x_1^2 + a_7x_2^2 + a_9x_3^2 + 2a_1x_0x_1 + 2a_2x_0x_2 + 2a_3x_0x_3 + 2a_5x_1x_2 + 2a_6x_1x_3 + 2a_8x_2x_3$$

where $a_0, a_1, \dots, a_9$ are linear forms in the variables of W. Direct computation shows that a quadric in the web q is annihilated by the equations $\partial_0\partial_2 - \partial_1^2, \partial_0\partial_3 - \partial_1\partial_2, \partial_1\partial_3 - \partial_2^2$ if and only if $a_2 = a_4, a_3 = a_5, a_6 = a_7$. It follows that the web q is orthic to the twisted cubic C iff its Hessian matrix has shape

$$\begin{pmatrix} b_0 & b_1 & b_2 & b_3 \\ b_1 & b_2 & b_3 & b_4 \\ b_2 & b_3 & b_4 & b_5 \\ b_3 & b_4 & b_5 & b_6 \end{pmatrix}$$

where $b_0, b_1, \ldots, b_6$ are linear forms in the variables of W, i.e. it is catalecticant. $\square$

It actually turns out that a 4×4 symmetric matrix with linear entries in $\mathbb{P}^3$ can be represented in two different ways as a catalecticant matrix. Namely

Proposition 3.2 *There are exactly two twisted cubic curves whose defining quadrics are annihilated by a general web of quadrics in* $\mathbb{P}^3$.

Proof. We use the same notation as in the proof of the previous proposition. Namely, let $q : W^* \hookrightarrow \mathrm{Sym}_2 V$ denote a general web of quadrics in $\mathbb{P}^3 = \mathbb{P}(V)$, and let $\check{\mathbb{P}}^3 = \mathbb{P}(V^*)$ denote the dual space. We also choose coordinates as above such that $S = k[x_0, \ldots, x_r]$ and $T = k[\partial_0, \ldots, \partial_r]$ are the coordinate rings of $\mathbb{P}(V)$ and $\mathbb{P}(V^*)$, respectively. Let now $q' : U \subset W^* \hookrightarrow \mathrm{Sym}_2(V)$ be a general subnet. The variety $H(q')$ of twisted cubics in $\mathbb{P}^(V^*)$ whose defining quadratic equations are annihilated by the net q' is the geometric realization of a prime Fano threefold X_q of genus 12 (see [Muk1] and [Sch]). Via the apolarity pairing, $X_q \subset \mathrm{Gr}(\mathbb{P}^2, \mathbb{P}(U_q))$, where the annihilator $U_q = (q^\perp)_2 \subset \mathrm{Sym}_2 V^*$ is a 7-dimensional vector space. As a subvariety of the Grassmannian the Fano threefold X_q is the (codimension 9) common zero-locus of three sections of $\wedge^2 \mathcal{E}$, where $\mathcal{E}$ is the dual of the tautological subbundle on $\mathrm{Gr}(\mathbb{P}^2, \mathbb{P}(U_q))$, cf. [Muk1] or see [Sch, Theorem 5.1] for a complete proof.

The choice of a (general) subnet q' is equivalent to the choice of a (general) global section of $\mathcal{E}$. But $\mathcal{E}$ is a globally generated rank 3 vector bundle whose restriction $E = \mathcal{E}_{|X_q}$ has third Chern number 2, as an easy direct computation shows. By Kleiman's transversality theorem the (general) section of E corresponding to q' must vanish exactly at two (simple) points of X_q. These in turn correspond to two twisted cubic curves each of whose defining quadrics are annihilated not only by the net q', but by the whole web q (the zero locus of a section in $\mathcal{E}$ is the special Schubert cycle of subspaces lying in the hyperplane dual to the section). This concludes the proof. $\square$

Let $C \subset \mathbb{P}^6$ be a rational normal sextic curve, and let $S = \mathrm{Sec}(C) \subset \mathbb{P}^6$ be its secant variety. S has degree 10, since this is the number of nodes of a general projection of C to a plane. The homogeneous ideal of C is generated by the 2×2-minors of either a 3×5 or a 4×4 catalecticant matrix with linear entries, induced by splittings of $\mathcal{O}_{\mathbb{P}^1}(6)$ as a tensor product of two line bundles of strictly positive degree. Furthermore, it is known that the homogeneous ideal of $S = \mathrm{Sec}(C)$ is generated by the 3×3 minors of either of the above two catalecticant matrices (see [GP] or [EKS]).

For $\Pi = \mathbb{P}^3 \subset \mathbb{P}^6$ a general 3-dimensional linear subspace, the linear section $\Gamma = \mathrm{Sec}(C) \cap \Pi$ consists of 10 simple points in $\mathbb{P}^3$ defined by the 3×3 minors of a 4×4 symmetric (even catalecticant) matrix with linear entries in the variables of Π. Conversely, by Proposition 3.2 above, the set $\Gamma \subset \mathbb{P}^3$ of 10 nodes of a general quartic symmetroid in $\mathbb{P}^3$ arises always as a linear section of the secant variety of the rational normal curve in $\mathbb{P}^6$ (in two different ways). Moreover, it follows that Γ can also be defined by the 3×3-minors of each of two different 3×5 catalecticant matrices with linear entries in $\mathbb{P}^3$. Since the 2×2 minors of these catalecticant matrices generate an irrelevant ideal, we may apply [EiPo, Theorem 6.1] (see also [EiPo, Example 6.3] for more details) to obtain the following

Proposition 3.3 *(Coble) The Gale transform of the* 10 *nodes of a general quartic symmetroid in* $\mathbb{P}^3$ *are the points of intersection of two Veronese surfaces in* $\mathbb{P}^5$.

Remark 1) A more careful analysis of the preceding argument shows that the needed generality assumptions on the quartic symmetroid are satisfied if the quartic symmetroid is defined by a *regular* web of quadrics in $\mathbb{P}^3$, see [Cos, Definition 2.1.2].
2) Coble asserts in [Cob] that the converse to Proposition 3.3 should also be true, presumably under suitable generality assumptions. This also relates to the question mentioned in [EiPo] of describing when a collection of 10 points in $\mathbb{P}^3$ are determinantal.

4 Further results

One may now ask which intersection numbers can actually occur and in which geometric situation this can happen. We are far from having a complete answer to this question, but want to state a number of results in this direction.

We start by considering Veronese surfaces which intersect in 10 points. It is easy to find examples of surfaces X_1 and X_2 intersecting transversally in 10 points. For this, one can start with an arbitrary surface $X_1 \subset \mathrm{Gr}(\mathbb{P}^1, \mathbb{P}^3) \subset \mathbb{P}^5$. For a general automorphism φ of $\mathbb{P}^3$ the surface $X_2 = \varphi(X_1)$ intersects X_1 transversally by Kleiman's transversality theorem and since both X_1 and X_2 have the same cohomology class we have $X_1.X_2 = 10$. Actually, we have the following

Proposition 4.1 *Let X_1, X_2 be two Veronese surfaces in $\mathbb{P}^5$ intersecting in 10 points. Then $X_1 \cup X_2$ is contained in a hyperquadric Q and one of the following cases occurs:*

(i) *Q has rank 6 and X_1 and X_2 lie in the same cohomology class,*

(ii) *The rank of Q is 4 and $X_1 \cap X_2 \cap \operatorname{Sing} Q \neq \emptyset$,*

Proof. The existence of Q follows from Proposition 1.3. If Q has rank 6 then X_1 and X_2 must have the same class by the proof of Proposition 2.1. The case rank $Q = 5$ is excluded by Proposition 2.2 and the case of rank $Q \leq 4$ and $X_1 \cap X_2 \cap \operatorname{Sing} Q = \emptyset$ is excluded by the remark at the end of Section 2. We are now left to exclude the case where rank $Q = 3$ and $X_1 \cap X_2 \cap \operatorname{Sing} Q \neq \emptyset$. We will make use of the diagram and the computations at the end of the proof of Proposition 2.4.

For a Veronese surface $X \subset Q$ with rank $Q = 3$ the class of $\tilde{X}$ in $\tilde{Q}$ equals $2H^2$. The fibres of the map $\pi|_{\tilde{X}} : \tilde{X} \to C \cong \mathbb{P}^1$ are conics and hence this linear system is a subsystem of $|2l|$ on $X \cong \mathbb{P}^2$, i.e. contained in some system of the form $|2l - \sum \alpha_i P_i|$. We have $(2l - \sum \alpha_i E_i)^2 = 4 - \sum \alpha_i^2 = 0$. This implies either $\alpha_1 = \ldots = \alpha_4 = 1$ or $\alpha_1 = 2$. But by the argument in the proof of Proposition 2.4 we already know that X intersects the vertex of Q in a finite non-empty set of points. Since the fibres of the map $\pi|_{\tilde{X}}$ are conics the first case can only occur if we have the linear system of conics through 4 points in general position. In this case $\tilde{X}$ is mapped to a $\mathbb{P}^1$ and the general fibre is an irreducible conic, which contradicts what we have. This implies that the linear system is given by $|2l - 2P|$, in particular X meets the vertex of Q in exactly one point P and that this intersection is not-transversal.

Returning to the case we have to exclude, we can assume that X_1 is given by the 2×2-minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_3 & x_4 \\ x_3 & x_4 & x_5 \end{pmatrix}$$

and so a typical rank 3 hyperquadric through X_1 is $Q = \{x_0x_3 - x_1^2 = 0\}$. The vertex V of Q is the plane $\{x_0 = x_1 = x_3 = 0\}$. The intersection of X_1 and V is then defined by the ideal (x_2x_4, x_2^2, x_4^2), i.e. the first infinitesimal neighborhood of P. The same holds for the second Veronese surface $X_2 \subset Q$. Since both surfaces $\tilde{X}_1$ and $\tilde{X}_2$ have class $2H^2$ on $\tilde{Q}$, it follows that $\tilde{X}_1$ and $\tilde{X}_2$ must meet in 6 other points (different of P). If these points are all in different fibers of π, then X_1 and X_2 meet in a (non-reduced) scheme of length $9(= 6 + 3)$ and thus we got a contradiction. Otherwise, since all the fibers of $\pi|_{\tilde{X}_i}$ are conics (whose images in $\mathbb{P}^5$ already go through the fixed point P), the 2-planes spanned by the fibres of $\pi|_{\tilde{X}_1}$ and $\pi|_{\tilde{X}_2}$ over some point in C must coincide. But then, by Bezout, the two conic fibres must intersect in 4 points, from which we conclude $X_1.X_2 \geq 11$, again impossible. This concludes the proof of the proposition. □

Remark As corollary of the proof of the previous proposition, we find out that two Veronese surfaces in $\mathbb{P}^5$ may intersect in a non-reduced scheme of length 9. This is indeed the case, for two (general) Veronese surfaces X_1 and X_2 lying on a rank 3 hyperquadric Q such that $X_1 \cap X_2 \cap \operatorname{Sing} Q$ consists of a single point P. As above, both surfaces cut out the first infinitesimal neighborhood of P on the vertex of the hyperquadric Q and meet further in 6 simple points. In particular their intersection is non-reduced.

However, we show that

Proposition 4.2 *Two Veronese surfaces in* $\mathbb{P}^5$ *cannot intersect transversally in* 9 *points.*

Proof. Assume that there exist Veronese surfaces X_1 and X_2 in $\mathbb{P}^5$ such that $W = X_1 \cap X_2$ is a reduced set of length 9. We consider again W as a subscheme of $X_1 \cong \mathbb{P}^2$ and we recall from Lemma 1.1 that $h^1(\mathcal{I}_W(4)) = 0$. Equivalently, this means that the linear system $\delta := |4l - W|$ on $X_1 \cong \mathbb{P}^2$ has projective dimension 5. It will then be enough to show that the linear system $|4l - W|$ of plane quartics through W contains a smooth curve C, since in this case the restriction of $|4l - W|$ defines, via taking the residual intersection, a $g_7^{4+\varepsilon}$ on C with $\varepsilon \geq 0$, thus contradicting Clifford's theorem.

In order to show the existence of such a C we consider the surface S given by blowing up the set W on X_1. It will be enough to check that the linear system $|4l - W|$ is base point free and defines a morphism $S \to S' \subset \mathbb{P}^5$ whose image S' has no worse than isolated singularities. We want to do this using Reider's theorem (see [BS, Theorem 2.1] in characteristic 0, and [SB], [Nak], [Ter, Theorem 2.4] in positive characteristic) and for this purpose we write $4l - W = L + K_\delta$ where $K_\delta = -3l + W$ is the canonical divisor on S while $L = 7l - 2W$. In order to apply a Reider type theorem we need to check that $L^2 \geq 9$ and that L is nef and big. The first is clear since $L^2 = 49 - 36 = 13$. We do not know that L is nef and big, but in the proof of Reider's theorem (as in the proof of its positive characteristic counterparts, cf [Ter]) this assumption is only used to conclude that $h^1(K_\delta + L) = 0$, which we already know since $h^1(\mathcal{I}_W(4)) = 0$ by Lemma 1.1.

We may now argue as follows. If $|4l - W| = |K_\delta + L|$ is not base-point free, respectively very ample, then there exists a curve a D such that $L - 2D$ is $\mathbb{Q}$-effective and such that

$$D^2 \geq L.D - k - 1$$

where $k = 0$, respectively 1. We write $D = al - \sum_{i=1}^{9} b_i E_i$. Since $L - 2D$ must be $\mathbb{Q}$-effective we see immediately that $a \leq 3$. For $a = 3$ we obtain

$$g - \sum_{i=1}^{9} b_i^2 \geq 21 - 2\sum_{i=1}^{9} b_i - k - 1$$

respectively

$$\sum_{i=1}^{9} b_i(b_i - 1) \leq -2 + k$$

which gives a contradiction. For $a = 1, 2$ the same calculation gives

$$\sum_{i=1}^{9} (b_i - 1)^2 \leq 4 + k \qquad (a = 1),$$

respectively

$$\sum_{i=1}^{9} (b_i - 1)^2 \leq k.$$

On the other hand, since the quadrics through X_2 cut out X_2, then at most 4 points of $W \subset \mathbb{P}^2$ can be collinear and at most 8 points of W can lie on the same conic. This shows that $|4l - W|$ is base point free on S and that S' has at most isolated singularities, which is our claim, and this concludes the proof of the proposition. □

Remark One may construct pairs of Veronese surfaces on suitable smooth or nodal cubic hypersurfaces in $\mathbb{P}^5$ which meet in 1, 2, 3, 5 or 6 simple points. It is also possible to check in *Macaulay* [Mac] that if $X \subset \mathbb{P}^5$ is a Veronese surface and φ is a general linear automorphism of $\mathbb{P}^5$ fixing $m \in \{1, 2, 3, 5\}$ (general) points on X, then X and $\varphi(X)$ meet exactly at those m points.

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