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# Robust exponential Little-Hopfield network storage

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## Abstract

The Little-Hopfield network is an auto-associative computational model of neural memory storage and retrieval. This model is known to robustly store collections of randomly generated binary patterns as stable-states of the network dynamics. However, the number of binary memories so storable scales linearly in the number of neurons, and it has been a long-standing open problem whether robust exponential storage of binary patterns was possible in such a network memory model. In this note, we design simple families of Little-Hopfield networks that provably solve this problem affirmatively. As a byproduct, we produce a set of novel (non-linear) binary codes with an efficient, highly parallelizable denoising mechanism.<sup>1</sup>

**1. Introduction.** Inspired by early work of McCulloch-Pitts [1] and Hebb [2], the Little-Hopfield model [3, 4] is a distributed neural network architecture for binary memory storage and denoising. In [4], Hopfield showed experimentally, using the *outer-product learning rule* (OPR), that  $.15n$  binary patterns (generated uniformly at random) can be robustly stored in such an  $n$ -node network if some fixed percentage of errors in a recovered pattern were tolerated. Later, it was verified that this number was a good approximation to the actual theoretical answer [5]. However, pattern storage without errors in recovery using OPR is provably limited to  $n/(4 \log n)$  patterns [6, 7]. Since then, improved methods to fit Little-Hopfield networks more optimally have been developed [8, 9, 10, 11]. Independent of the method, however, arguments of Cover [12] show that the number of (randomly generated) patterns storable in a Little-Hopfield network with  $n$  neurons is at most  $2n$ .

Nonetheless, theoretical and experimental evidence suggest that Little-Hopfield networks usually have exponentially many stable-states (i.e., fixed-points of the dynamics). For instance, choosing weights for the model randomly (from a normal distribution) produces an  $n$ -node network with  $\approx 1.22^n$  fixed-points asymptotically [13, 14, 15]. However, a stored pattern corrupted by only a few bit errors does not typically converge under the network dynamics to the original.

Another limitation of random networks is that stable-states are difficult to determine from the network parameters. In [16], a Little-Hopfield network with identical weights was shown to have exponential storage on  $2n$  nodes, the stored collection consisting of binary vectors with exactly half of their bits equal. Thus, it is possible to design a network with a prescribed exponential number

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$\binom{2n}{n} \approx \frac{4^n}{\sqrt{\pi n}}$  of patterns. However, such a network is not able to denoise a single bit of corruption. In particular, this collection of memories is not stored robustly (a concept we define momentarily).

Very recently, more sophisticated (non-binary) discrete networks have been developed [17, 18] that give exponential memory storage. However, the storage in these networks is not known to be robust. Moreover, determining or prescribing the network parameters for storing these exponentially many memories is non-trivial (e.g., the ideas involve expander codes and solving linear equations over  $\mathbb{Z}$ ).

In this work, we design elementary Little-Hopfield networks that robustly store an exponential number of binary patterns. Our main result is the following.

**Theorem 1** *For each integer  $v = 2k > 3$ , there is an easily constructed Little-Hopfield network on  $n = \binom{v}{2}$  nodes that robustly stores an exponential number  $\binom{v}{k} \approx \frac{2^{\sqrt{2n} + \frac{1}{4}}}{n^{1/4}\sqrt{\pi}}$  of memories.*

The outline of this note is as follows. In Section 2, we provide technical background and context, defining the concepts needed to state and prove our main results. Proofs are given in Section 4 and Section 5 closes with a discussion of potential applications.

**2. Background.** A Little-Hopfield network  $\mathcal{H} = (\mathbf{J}, \theta)$  on  $n$  nodes (e.g. neurons)  $\{1, \dots, n\}$  consists of a real symmetric weight matrix  $\mathbf{J} = \mathbf{J}^\top \in \mathbb{R}^{n \times n}$  with zero diagonal and a threshold vector  $\theta \in \mathbb{R}^n$ . The possible states of the network are all length  $n$  binary strings  $\{0, 1\}^n$ , which we represent as binary column vectors  $\mathbf{x} = (x_1, \dots, x_n)^\top$ , each  $x_i \in \{0, 1\}$  indicating the state  $x_i$  of node  $i$ . Given any state  $\mathbf{x}$ , one (asynchronous) update of the dynamics on  $\mathbf{x}$  consists of replacing each  $x_i$  in  $\mathbf{x}$  (in consecutive order starting with  $i = 1$ ) with the value

$$x_i = H(\mathbf{J}_i^\top \mathbf{x} - \theta_i) = H\left(\sum_{j \neq i} J_{ij} x_j - \theta_i\right). \quad (1)$$

$\mathbf{J}_i$  is the  $i$ th column of  $\mathbf{J}$  and  $H$  is the Heaviside function:  $H(r) = 1$  if  $r > 0$ ;  $H(r) = 0$  if  $r \leq 0$ .

The energy  $E_{\mathbf{x}} = E_{\mathbf{x}}(\mathbf{J}, \theta)$  of a binary pattern  $\mathbf{x}$  in a Little-Hopfield network is

$$E_{\mathbf{x}}(\mathbf{J}, \theta) := -\frac{1}{2} \mathbf{x}^\top \mathbf{J} \mathbf{x} + \theta^\top \mathbf{x} = -\sum_{i < j} x_i x_j J_{ij} + \sum_{i=1}^n \theta_i x_i, \quad (2)$$

equal to the energy for an Ising spin glass probabilistic model from statistical physics [19, 20]. In fact, the dynamics of Little-Hopfield networks can be seen as 0-temperature sampling of this energy.

A fundamental property of Little-Hopfield networks is that asynchronous dynamical updates (1) do not increase the energy (2). In particular, one can show that after a finite number of updates, any initial state  $\mathbf{x}$  converges to a fixed-point  $\mathbf{x}^*$  (also called *stable-state* or *stored memory*) of the dynamics; that is,  $x_i^* = H(\mathbf{J}_i^\top \mathbf{x}^* - \theta_i)$  for each  $i = 1, \dots, n$ . Given a binary pattern  $\mathbf{x}$ , we say more strongly that it is a *strict local minimum* if every  $\mathbf{x}'$  with exactly one bit different from  $\mathbf{x}$  has a strictly larger energy:  $0 > E_{\mathbf{x}} - E_{\mathbf{x}'} = (\mathbf{J}_i^\top \mathbf{x} - \theta_i)(1 - 2x_i)$ , where  $x_i$  is the bit that differs between  $\mathbf{x}$  and  $\mathbf{x}'$ . One can check that if  $\mathbf{x}$  is a strict local minimum it is also fixed-point of the dynamics.

In abstract algebra, a *group* is a set  $G$  with a multiplication (or product)  $a \circ b$  between elements  $a, b \in G$  satisfying the following three assumptions. We have (i) *associativity of the product*:  $(a \circ b) \circ c = a \circ (b \circ c)$  for all  $a, b, c \in G$ ; (ii) a *multiplicative identity*: there is a unique element  $1 \in G$  with  $a \circ 1 = 1 \circ a = a$  for all  $a \in G$ ; and (iii) *existence of inverses*: for all  $a \in G$ , there exists  $a^{-1} \in G$  with  $a \circ a^{-1} = a^{-1} \circ a = 1$ . Groups are basic but fundamental objects.

Fix a positive integer  $v$ . The set of bijections from the integers  $V = \{1, \dots, v\}$  to themselves are called the *permutations*  $S_v$  of  $V$ . The set of permutations  $S_v$  has size  $v! = v \cdot (v-1) \cdots 1$  and forms a group with composition of functions as the product. Sometimes permutations are displayed with two rows that indicate the bijection. For instance, the permutation  $\sigma \in S_5$  mapping the numbers  $(1, 2, 3, 4, 5)$  bijectively to  $(2, 1, 5, 3, 4)$  and its inverse  $\sigma^{-1}$  can be represented:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{pmatrix}, \quad \sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 4 & 5 & 3 \end{pmatrix}. \quad (3)$$

A permutation  $\sigma \in S_n$  of the  $n$ -nodes of a Little-Hopfield network  $\mathcal{H} = (\mathbf{J}, \theta)$  gives rise to another network  $\sigma\mathcal{H} = (\sigma\mathbf{J}, \sigma\theta)$ , where  $\sigma\mathbf{J}$  is the matrix obtained from  $\mathbf{J}$  by permuting both its rows and columns by  $\sigma$ , and where  $\sigma\theta$  is  $\theta$  also permuted by  $\sigma$ .

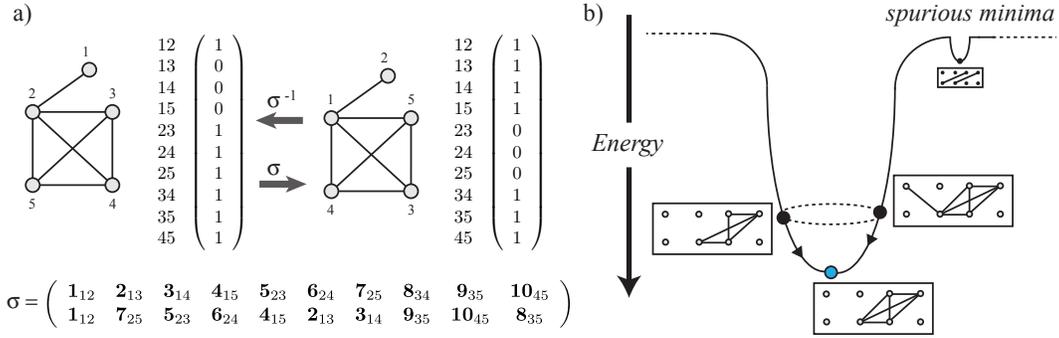


Figure 1: **a) Permutations acting on graphs.** A graph on  $v = 5$  vertices is encoded as a binary vector of length  $n = \binom{5}{2} = 10$ . Applying the permutation  $\sigma$  in (3) to the vertices  $V = \{1, 2, 3, 4, 5\}$  induces a permutation in  $S_n$  of the vector encoding the graph, which we also denote  $\sigma$ . **b) Energy landscape** of a Little-Hopfield network depicting the robust storage of all 4-cliques in graphs on  $v = 8$  vertices. The network dynamics sends a graph that is almost a clique to a graph with smaller energy, until converging to a 4-clique.

A *graph* on  $v$  vertices  $V = \{1, \dots, v\}$  is a set  $E$  of (unordered) pairs of vertices, called the *edges* of the graph. We identify graphs on  $v$  vertices as binary vectors  $\mathbf{x}$  of length  $n = \binom{v}{2} = \frac{v(v-1)}{2}$ . A coordinate  $x_e$  of  $\mathbf{x}$  is indexed by an edge  $e = \{i, j\}$  ( $i < j$ ), and is one or zero depending on whether  $e$  is contained in the edges of the graph or not (respectively). For simplicity, we list the coordinates in  $\mathbf{x}$  lexicographically (i.e., the dictionary order). For  $3 \leq k \leq v$ , define a *k-clique* to be a graph on  $v$  vertices that has edges between each pair of a set of  $k$  vertices, but no other edges. There are  $\binom{v}{k}$  graphs on  $v$  vertices that are *k-cliques*. The *complete graph*  $K_v$  on  $v$  vertices is a *v-clique*.

Relabeling the vertices of a graph is the same as applying a permutation  $\sigma \in S_v$  to them. This, in turn, induces a relabeling or permutation of the edges of the graph, which is realized as a permutation of the vector  $\mathbf{x}$  representing it. Note that any permutation of  $V$  for a *k-clique* gives another *k-clique*. Given a Little-Hopfield network with weight matrix  $J$ , a permutation  $\sigma$  on the vertices  $V = \{1, \dots, v\}$  induces a permutation of the edges of a graph, defining a new weight matrix  $\sigma J$ , which is the rows and columns of  $J$  permuted accordingly.

**3. Main results.** The storage networks we propose are Little-Hopfield networks with states identified as simple graphs on  $v$  vertices. In this case, entries of weight matrices  $J \in \mathbb{R}^{n \times n}$ ,  $n = \binom{v}{2}$ , are indexed lexicographically by pairs of edges  $e, f$  in  $K_v$ , the complete graph on  $v$  vertices.

We first formalize a notion of robust storage for precise statements of our main results. Equip the state space  $\{0, 1\}^n$  with the Hamming distance, the minimal number of bit flips between two binary strings. For an integer  $r \geq 1$ , we say that state  $\sigma^*$  is *r-stable* if it is an attractor for all states within Hamming distance  $r$  of  $\sigma^*$ . Thus, if a state  $\sigma$  is *r-stably* stored, the network is guaranteed to converge to  $\sigma$  when exposed to any corrupted version not more than  $r$  bit flips away from  $\sigma$ . This is a fairly restrictive definition since it might be useful for states to have stability “on average”, or for “most randomly corrupted versions”. We make this last concept rigorous by introducing *p-corruption*.

*Definition 1 (p-corruption):* Let  $\sigma \in \{0, 1\}^n$  be a state,  $p \in [0, 1]$ . The *p-corruption* of  $\sigma$  is the random pattern  $\Sigma_p$ , obtained by replacing each  $x_i$  by  $1 - x_i$  with probability  $p$  independently.

We now define  $\alpha$ -robustness for a set of states  $S$ , which says that for each  $\sigma \in S$ , with high probability, the network is guaranteed to recover  $\sigma$  when exposed to a *p-corruption* of  $\sigma$  for all  $p < \alpha$ .

*Definition 2 ( $\alpha$ -robust):* Let  $\mathcal{H}$  be a Little-Hopfield network,  $S \subset \{0, 1\}^n$ , and fix  $\epsilon \in [0, 1]$ . For  $p \in [0, 1]$  and  $\sigma \in S$  let  $\Sigma_p$  be the *p-corruption* of  $\sigma$ . The robustness  $\alpha(S, \epsilon)$  of  $S$  in  $\mathcal{H}$  is

$$\alpha(S, \epsilon) := \operatorname{argmax}_{p \in [0, 1]} \left\{ \min_{\sigma \in S} \mathbb{P}(\sigma \text{ is an attractor of } \Sigma_p) \geq 1 - \epsilon \right\}.$$

Finally, the robustness index of a sequence of states in a corresponding sequence of Little-Hopfield networks is their asymptotic  $\alpha$ -robustness. In this sense, the networks mentioned in the introduction do not have robust storage (that is, their robustness index is 0) because the number of bits of corruption tolerated in memory recovery does not increase with the number of nodes.

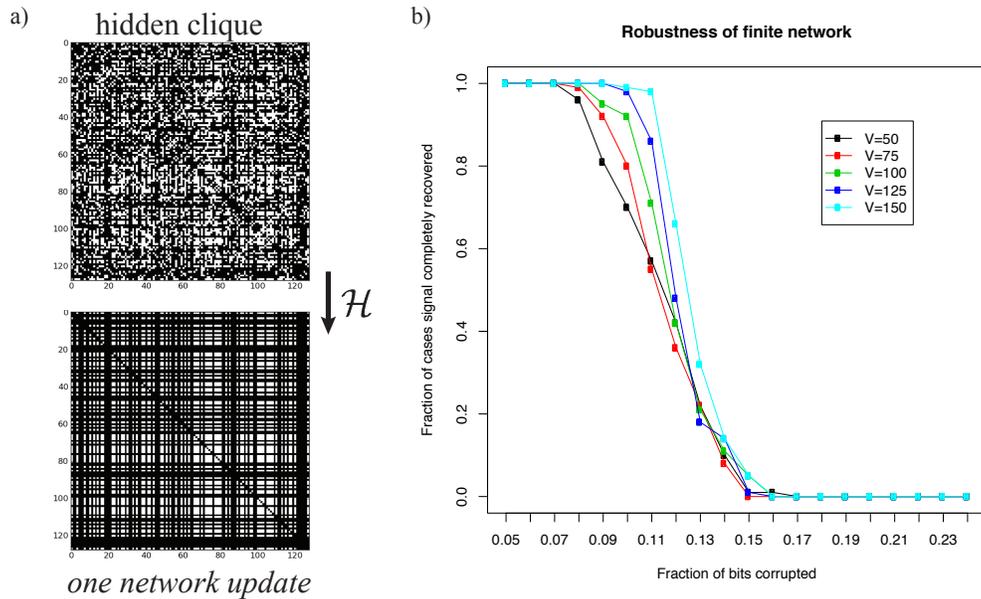


Figure 2: **a) Hidden clique.** A 64-clique on 128 vertices is described by its *adjacency matrix*  $A$  (image on bottom left). The matrix  $A = (a_{ij})_{i,j=1}^{128}$  has  $a_{ij} = 1$  or 0 depending on whether  $\{i, j\}$  are both in the clique or not, respectively. The upper left image represents the graph in **a** with 20% of its bits flipped at random. All  $\binom{128}{64} \approx 10^{37}$  64-cliques are exactly recovered (with high probability) after one network update. **b) Robustness of networks** with  $v = 50, 75, 100, 125$  and 150 vertices with parameters given in our main result.

*Definition 3 (robustness index):* Let  $(\mathcal{H}_n)$  be a sequence of Little-Hopfield networks and  $(S_n)$  be a sequence of pattern collections. We say that the sequence  $S_n$  is *robustly stored* by  $\mathcal{H}_n$  with *robustness index*  $\alpha > 0$  if

$$\liminf_{\epsilon \rightarrow 0, n \rightarrow \infty} \alpha(S_n, \epsilon) \rightarrow \alpha.$$

Note that the  $p$ -corruption of a state differs from the original by  $np$  bit flips on average. Clearly the larger  $p$  is, the more difficult to recover the original binary pattern.

**Theorem 2** *For each positive integer  $k > 3$ , there exists an explicit two-parameter family of Little-Hopfield networks on  $n = \binom{2k}{2}$  nodes that stores  $\binom{2k}{k}$  known stable states ( $k$ -cliques on  $2k$  vertices). In particular, there exists an explicit sequence of Little-Hopfield networks  $(\mathcal{H}_n)$  such that asymptotically,  $\mathcal{H}_n$  robustly stores approximately  $\frac{2^{\sqrt{2n} + \frac{1}{4}}}{n^{1/4} \sqrt{\pi}}$  known stable states with robustness index  $\alpha = \frac{1}{2}$ .*

While the robustness index is defined to be an asymptotic quantity, we can construct finite-sized Little-Hopfield networks which store  $k$ -cliques robustly. In Figure 2, we demonstrate that the exponential number of stored cliques in our networks have large basins of attraction. For each vertex size  $v = 50, 75, 100, 125$  and 150 (with  $k = 25, 37, 50, 63$  and 75), we constructed a Little-Hopfield network storing all  $k$ -cliques as fixed-points of the dynamics using the parameters identified in Corollary 1. Each such  $k$ -clique is represented as a binary vector of length  $(2k - 1)k$ . In each experiment, we chose a  $k$ -clique uniformly at random, corrupted each of the  $\binom{v}{2}$  potential edges independently with probability  $p$ , and ran the network dynamics until convergence. We performed 100 such experiments, and plotted the fraction of the 100 cliques that were correctly recovered (exactly) as a function of  $p$ . For example, a network with  $v = 100$  vertices robustly stores  $\binom{100}{50} \approx 10^{29}$  memories (i.e., all 50-cliques in a 100-node graph) using binary vectors of length 4950, each having  $\binom{50}{2} = 1225$  nonzero coordinates. The figure shows that a 50-clique represented with 4950 bits is recovered by the dynamics  $\approx 90\%$  of the time after flipping on average  $\frac{1}{10} \cdot 4950 = 495$  at random.

Theorem 2 says that we may store all cliques of a certain fixed size in a Little-Hopfield network. A natural question is whether a range of cliques are so storable as fixed-points of a single network.

**Theorem 3** For each integer  $v > 3$ , there is a Little-Hopfield network on  $n = \binom{v}{2}$  nodes that stores all  $\approx 2^v(1 - e^{-Cv})$   $k$ -cliques in the range  $R = [\frac{1}{D+2}v, \frac{3D+2}{2(D+2)}v]$  as strict local minima for constants  $C \approx .002$ ,  $D \approx 13.93$ . Moreover, this is the best possible range for any Little-Hopfield network.

We close this section by sketching the main proof ideas. We first show that there is a Little-Hopfield weight matrix storing all  $k$ -cliques in some range if and only if there is one which has a simple 3-parameter structure. Note that the set of all  $J$  storing a given set of binary patterns as strict local minima is the interior of a (possibly empty) convex polyhedron.

As discussed above, the symmetric group  $S_v$  acts on weight matrices  $J$ . Consider the average of  $J$  over the group of permutations:

$$J^* := \frac{1}{d!} \sum_{\tau \in S_v} \tau \cdot J, \quad \theta^* := \frac{1}{d!} \sum_{\tau \in S_v} \tau \cdot \theta. \quad (4)$$

The matrix  $J^*$  in (4) is *invariant* under the action of  $S_v$ ; that is, we have

$$\tau J^* = \frac{1}{v!} \sum_{\sigma \in S_v} \tau \sigma J = J^*,$$

since the function from  $S_v$  to itself mapping  $\sigma \mapsto \tau \sigma$  (for any fixed  $\tau \in S_v$ ) is a bijection.

It is straightforward to verify that acting by a permutation on a Little-Hopfield network that stores all  $k$ -cliques as strict local minima will preserve that property. And since the set of all such networks is convex, the convex combination  $J^*$  in (4) stores all  $k$ -cliques as strict local minima if  $J$  does. One now observes that  $J^*$  has only 3 free parameters, and the remainder of the argument consists of optimizing these parameters to determine networks that store cliques robustly.

**4. Proofs of main results:** Fix positive integers  $v \geq k > 3$ , and set  $n = \binom{v}{2}$ . The set  $\{0, 1\}^n$  of states of a Little-Hopfield network on  $n$  nodes are in bijection with subgraphs of  $K_v$ , where  $\sigma \in \{0, 1\}^n$  encodes the graph with edge  $e$  if and only if  $\sigma_e = 1$ . Let  $\theta(z) = z \mathbf{1}^\top$ , where  $\mathbf{1}$  is the all-ones vector in  $\mathbb{R}^n$ . Consider the following two-parameter family of symmetric weight matrices

$$J(x, y)_{ef} = \begin{cases} x & \text{if } |e \cap f| = 1 \\ y & \text{if } |e \cap f| = 0 \end{cases} \quad (5)$$

for some  $x, y \in \mathbb{R}$ , where  $|e \cap f|$  is the number of vertices that edges  $e$  and  $f$  share.

**Lemma 1** Fix  $k > 3$  and  $1 \leq r < k$ . The Little-Hopfield network  $(J(x, y), \theta(z))$  stores all  $k$ -cliques as  $r$ -stable states if and only if the parameters  $x, y, z \in \mathbb{R}$  satisfy the following inequalities

$$\begin{bmatrix} -4(k-2) + 2r & -(k-2)(k-3) & 2 \\ -4(k-2) & -(k-2)(k-3) - 2r & 2 \\ 2(k-1) + 2r & (k-1)(k-2) & -2 \\ 2(k-1) & (k-1)(k-2) - 2r & -2 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} < \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Furthermore, a pattern within Hamming distance  $r$  of a  $k$ -clique converges after one iteration of the network dynamics.

*Proof:* For fixed  $r$  and  $k$ -clique  $\sigma$ , there are  $2^r$  possible patterns within Hamming distance  $r$  of  $\sigma$ . Each of these pattern defines a pair of linear inequalities on the parameters  $x, y, z$ . However, only the inequalities from the following two extreme cases are active constraints.

- Case 1:  $r$  edges are added to a node  $i$  not in the clique.
- Case 2:  $r$  edges in the clique with a common node  $i$  are removed.

In Case 1, consider the edges of the form  $(i, j)$  for all nodes  $j$  in the clique. Such an edge has  $r + k - 1$  neighboring edges and  $\binom{k-1}{2}$  non-neighboring edges. Thus, such an edge will be labeled as 0 after one network update if and only if  $x, y$  satisfy:

$$2(r + k - 1)x + (k - 1)(k - 2)y \leq 1. \quad (6)$$

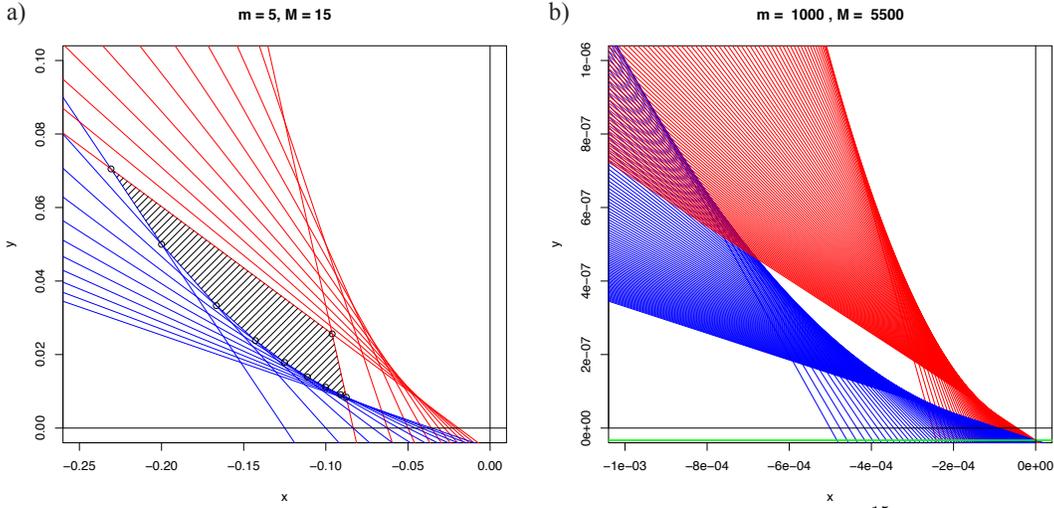


Figure 3: **a) Feasible exponential storage.** The shaded region is the feasible polytope  $\bigcap_{k=5}^{15} \mathcal{I}_k$  for network parameters giving clique storage for  $5 \leq k \leq 15$ . Black points are its vertices, the red  $R_k$  and blue  $B_k$  lines are linear constraints. **b) Feasible region for range of  $k$ .** Lines  $R_k$  (red) and  $B_k$  (blue) for  $1000 \leq k \leq 5500$ .

Edges of the form  $(j, j')$  for all nodes  $j, j'$  in the clique have  $2(k-2)$  neighbors and  $\binom{k-2}{2} + r$  non-neighbors. Thus they impose the following linear constraint:

$$4(k-2)x + ((k-2)(k-3) + 2r)y > 1. \quad (7)$$

In Case 2, the edges at risk of being mislabeled are edges of the form  $(i, j)$  for all nodes  $j$  in the clique. Such an edge has  $2(k-2) - r$  neighbors and  $\binom{k-2}{2}$  non-neighbors. This results in another linear constraint for  $x, y$ :

$$2(2k - r - 4)x + (k-2)(k-3)y > 1. \quad (8)$$

Consider edges of the form  $(\bar{i}, j)$  for all nodes  $\bar{i} \neq i$  in the clique, and  $j$  not in the clique. Assuming  $r < k-1$ , such an edge has at most  $k-1$  neighbors and  $\binom{k-1}{2} - r$  non-neighbors. Thus it imposes the following constraint

$$2(k-1)x + ((k-1)(k-2) - 2r)y \leq 1. \quad (9)$$

The equations of Lemma 1 cut out a cone in  $\mathbb{R}^3$ . Thus without loss of generality, we can assume either  $z = 1/2$  or  $z = 0$ . Suppose  $z = 0$  and assume  $y = -1$ . This forces  $x > 0$ , and the second and fourth constraints are dominated by the first and third. Thus we need  $x$  that solve

$$(4k - 2r - 8)x > (k-2)(k-3) \quad \text{and} \quad (2k + 2r - 2)x < (k-1)(k-2).$$

To see how large  $r$  can be relative to  $k$ , we examine at the asymptotics as  $k \rightarrow \infty$  and approximate each coefficient of the matrix by its first-order term. Suppose  $r = ck$  for  $c > 0$ . For large  $k$ , we have

$$(4 - 2c)kx > k^2 + O(k) \quad \text{and} \quad (2 + 2c)kx < k^2 + O(k).$$

Thus we need  $\frac{k}{2(2-c)} + O(1) < x < \frac{k}{2(1+c)} + O(1)$ , which is feasible for all  $c < 1/2$ , and infeasible for all  $c > 1/2$ . Thus we can expect to set  $r = \lfloor k/2 \rfloor$ . By direct computation, we obtain:

**Corollary 1** Fix  $z = 0$  and  $y = -1$ . For  $k = 2r + 1$ , the set of feasible solutions in Lemma 1 is

$$\frac{2r^2 - 3r + 1}{3r - 2} < x < \frac{2r - 1}{3},$$

which is non-empty for  $r \geq 1$ . In particular,  $x = \frac{2r-1.5}{3} = \frac{k-2.5}{3}$  is a solution for all  $r > 1, k > 3$ .

Thus, setting  $x = \frac{k-2.5}{3}, y = -1, z = 0$ , we get a family of Little-Hopfield networks in which all  $k$ -cliques are stored as  $\lfloor k/2 \rfloor$ -stable states. This already proves exponential storage in Theorem 2.

We now tune the parameters  $(x, y, z)$  to guarantee robust storage of all  $k$ -cliques. As before, fix  $y = -1, z = 0$ . For  $p \in (0, 1)$ , let  $\Sigma_p$  be a  $p$ -corruption of a  $k$ -clique chosen uniformly at random. For each node  $i$ , let  $i_{in}, i_{out}$  denote the number of edges from  $i$  to other clique and non-clique nodes, respectively. We need the following lemma.

**Lemma 2** *Let  $Y$  be an  $n \times n$  symmetric matrix with zero diagonal, where  $Y_{ij}$  are i.i.d Bernoulli( $p$ ) random variables for  $i < j$ . For each  $i = 1, \dots, n$ , let  $Y_i = \sum_j Y_{ij}$  be the  $i$ -th row sum. Let  $M_n = \max_i Y_i$ , and  $m_n = \min_i Y_i$ . Then for any constant  $c > 0$ ,*

$$\mathbb{P}(|m_n - np| > c\sqrt{n} \log(n)) \rightarrow 0, \quad \mathbb{P}(|M_n - np| > c\sqrt{n} \log(n)) \rightarrow 0 \text{ as } n \rightarrow \infty.$$

*In particular, we have  $|m_n - np|, |M_n - np| = o(\sqrt{n} \log(n))$ .*

*Proof:* Fix a constant  $c > 0$ . By Bernstein's inequality [21], for each  $i$  and for any  $\epsilon > 0$ ,

$$\mathbb{P}(Y_i - np > n\epsilon) \leq \exp\left(-\frac{n\epsilon^2}{2 + 2\epsilon/3}\right).$$

Applying the union bound and choosing  $\epsilon = \frac{c \log(n)}{\sqrt{n}}$ , we have

$$\mathbb{P}(\max_i Y_i - np > n\epsilon) \leq \exp\left(-\frac{n\epsilon^2}{2 + 2\epsilon/3} + \log(n)\right) \leq \exp\left(-\frac{\log(n)^2}{3} + \log(n)\right),$$

and the last bound converges to 0 as  $n \rightarrow \infty$ , proving the claim for  $M_n$ . Since the distribution of  $Y_i$  is symmetric about  $np$ , a similar inequality holds for  $m_n$ . ■

**Corollary 2** *Let  $M_{in} = \max_{i \in C_k} i_{in}$ ,  $m_{in} = \min_{i \in C_k} i_{in}$ ,  $M_{out} = \max_{i \notin C_k} i_{out}$ ,  $m_{out} = \min_{i \notin C_k} i_{out}$ , and  $M_{between} = \max_{i \notin C_k} i_{in}$ . Then  $M_{in} - k(1-p), m_{in} - k(1-p), M_{out} - kp, m_{out} - kp$ , and  $M_{between} - kp$  are all of order  $o(\sqrt{k} \log(k))$  as  $k \rightarrow \infty$  almost surely.*

Let  $T$  be the total number of edges in  $\Sigma_p$ . Then  $T$  is the sum of two independent binomials

$$\text{Binom}\left(\binom{k}{2}, 1-p\right) + \text{Binom}\left(\binom{2k}{2} - \binom{k}{2}, p\right).$$

For an edge  $e$ , let  $N(e)$  and  $\bar{N}(e)$  be the number of neighbors and non-neighbors of  $e$ , respectively. Note that  $\bar{N}(e) = T - N(e)$ . For every  $e$  in the clique, we have (with high probability)

$$N(e) \geq 2m_{in} + 2m_{out} \sim 2k + o(\sqrt{k} \log(k)) \quad \text{w.h.p.}$$

Since the standard deviation is of order  $k$ , by a union bound, we have

$$\bar{N}(e) \leq T - (2m_{in} + 2m_{out}) \sim k^2(1/2 + p) + O(k) \quad \text{w.h.p.}$$

To guarantee that  $\sigma_e = 1$  for all edges  $e$  in the clique after one update iteration, we need  $N(e)\sigma - \bar{N}(e) > 0$  so that

$$x > \left(\frac{1+2p}{4}\right)k + o(\sqrt{k} \log(k)). \quad (10)$$

Now let  $f$  be an edge with only one vertex in the clique. Then

$$N(f) \leq M_{in} + M_{out} + 2M_{between} \sim k(1+2p) + o(\sqrt{k} \log(k)) \quad \text{w.h.p.}$$

By the same argument as above, we have

$$\bar{N}(f) \geq T - (M_{in} + M_{out} + 2M_{between}) \sim k^2(1/2 + p) + O(k) \quad \text{w.h.p.}$$

To guarantee that  $\sigma_f = 0$  for all such edges  $f$  after one iteration, we need  $N(f)\sigma - \bar{N}(f) < 0$ . Thus we obtain the following constraint for  $x$ :

$$x < \frac{1}{2}k + o(\sqrt{k} \log(k)). \quad (11)$$

In particular, if  $p = p(k) \sim \frac{1}{2} - k^{\delta-1/2}$  for some small  $\delta \in (0, 1/2)$ , then taking  $x = x(k) = \frac{1}{2}k(1 - k^{\delta/2-1/4})$  would guarantee that for large  $k$ , both equations (10) and (11) are simultaneous

satisfied. In this case,  $\lim_{k \rightarrow \infty} p(k) = 1/2$ , and thus the family of two-parameter Little-Hopfield networks with  $x(k) = \frac{1}{2}k(1 - k^{\delta/2 - 1/4})$ ,  $y = -1$ ,  $z = 0$  has robustness index  $\alpha = 1/2$ . ■

*Proof of Theorem 3:* For large  $m, M$  and  $v$ , we have the approximations  $x_m \approx \frac{\sqrt{12}-4}{2m}$ ,  $x_M \approx \frac{-\sqrt{12}-4}{2M}$  for  $x_m, x_M$  defined in Theorem 4. Hence  $x_M - x_m < 0$  when  $M \lesssim \frac{2+\sqrt{3}}{2-\sqrt{3}}m \approx 13.9282m$ .

Note that  $\binom{v}{k}2^{-v}$  is the fraction of  $k$ -cliques in  $K_v$ , which is also the probability of a *Binom*( $v, 1/2$ ) equalling  $k$ . To store the most cliques, choose  $m = \frac{1}{1 + \frac{2+\sqrt{3}}{2-\sqrt{3}}} \approx \frac{1}{15}v$ . For large  $v$ , approximating the binomial distribution by a Gaussian, and then using Mill's ratio [22, p. 98] to approximate the tail of the Gaussian c.d.f, we see that the proportion of cliques storable tends to

$$\Phi\left(\frac{14}{15}\sqrt{v}\right) - \Phi\left(\frac{1}{15}\sqrt{v}\right) = 1 - 2\Phi\left(\frac{1}{15}\sqrt{v}\right) \approx 1 - \exp(-Cv)$$

for some constant  $C \approx \frac{1}{2 \cdot 15^2} \approx 0.002$ . The range appearing in Theorem 3 arises from rounding. ■

Theorem 3 is actually a corollary of the following more precise version of our range storage result.

**Theorem 4** Fix  $m$  such that  $3 \leq m < v$ . For  $M \geq m$ , there is a Little-Hopfield network that stores all  $k$ -cliques in the range  $[m, M]$  if and only if  $M$  solves the implicit equation  $x_M - x_m < 0$ , where

$$x_m = \frac{-(4m - \sqrt{12m^2 - 52m + 57} - 7)}{2(m^2 - m - 2)}, x_M = \frac{-(4M + \sqrt{12M^2 - 52M + 57} - 7)}{2(M^2 - M - 2)}.$$

In Figure 3 we plot  $\mathcal{I}_k$  for  $5 \leq k \leq 15$  and shade their intersections. Figure 3 shows a similar plot for  $1000 \leq k \leq 5500$ . Note the appearance of the smooth curve  $Q$  enveloping the family  $B_k$ .

**5. Discussion.** The Little-Hopfield network is a model of emergent neural computation [3, 4, 25]. Although originally intended as a circuit model for biological memory circuits, the local network dynamics can be thought of as implementing perceptual inference: by minimizing an energy function, the network infers the most probable memory conditioned on a noisy or corrupted version. (The term ‘‘local’’ here refers to the fact that an update (1) to a neuron only requires the feed-forward inputs from its neighbors.) This concept is in line with arguments of several researchers in theoretical neuroscience (e.g., [26]), and can be traced back to Helmholtz [27]. Additional support for this view stems from recent analyses of spike distributions in neural populations which have shown that their joint statistics can sometimes be well-described by the Ising model [28, 29, 30, 31]. The now demonstrated ability of these networks to store large numbers of patterns robustly suggests that the Little-Hopfield architecture should be revisited as a possible model of neural circuit computation.

Beyond neuroscience, the networks described in this note have potential implications for several algorithmic problems at the intersection of discrete mathematics, probability, computer science, and machine learning. For instance, a classical NP-complete problem is to determine large cliques in graphs, the so-called MAXCLIQUE problem. We have demonstrated here that when a clique is planted into an empty graph and then ‘‘hidden’’ by turning edges on and off at random, it is still possible to recover the original clique by converging the local dynamics of Little-Hopfield networks. See [23] for the most recent results on a problem of this nature. Our networks also gives rise to new approaches for constructing and working with binary codes. For instance, our networks are easily parallelizable and have similar robustness properties to the well-known optimal codes of Reed-Solomon [24], which use the mathematical machinery of polynomial rings over finite fields.

## References

- [1] W. McCulloch and W. Pitts, ‘‘A logical calculus of the ideas immanent in nervous activity,’’ *Bulletin of mathematical biology*, vol. 5, no. 4, pp. 115–133, 1943.
- [2] D. Hebb, ‘‘The organization of behavior. 1949,’’ *New York Wiley*, 2002.
- [3] W. Little, ‘‘The existence of persistent states in the brain,’’ *Mathematical Biosciences*, vol. 19, no. 1, pp. 101–120, 1974.
- [4] J. Hopfield, ‘‘Neural networks and physical systems with emergent collective computational abilities,’’ *Proceedings of the National Academy of Sciences of the United States of America*, vol. 79, no. 8, p. 2554, 1982.

- [5] D. Amit, H. Gutfreund, and H. Sompolinsky, “Statistical mechanics of neural networks near saturation,” *Annals of Physics*, vol. 173, no. 1, pp. 30–67, 1987.
- [6] G. Weisbuch and F. Fogelman-Soulié, “Scaling laws for the attractors of Hopfield networks,” *Journal de Physique Lettres*, vol. 46, no. 14, pp. 623–630, 1985.
- [7] R. McEliece, E. Posner, E. Rodemich, and S. Venkatesh, “The capacity of the Hopfield associative memory,” *Information Theory, IEEE Transactions on*, vol. 33, no. 4, pp. 461–482, 1987.
- [8] D. Wallace, “Memory and learning in a class of neural network models,” *Lattice Gauge Theory—A Challenge to Large Scale Computing*, 1986.
- [9] A. Bruce, A. Canning, B. Forrest, E. Gardner, and D. Wallace, “Learning and memory properties in fully connected networks,” in *AIP Conference Proceedings*, vol. 151, 1986, p. 65.
- [10] M. Jinwen, “The asymmetric hopfield model for associative memory,” in *Neural Networks, 1993. IJCNN’93-Nagoya. Proceedings of 1993 International Joint Conference on*, vol. 3. IEEE, 1993, pp. 2611–2614.
- [11] A. Anonymous, A. Anonymous, and A. Anonymous, “Efficient and optimal Little-Hopfield auto-associative memory storage using minimum probability flow,” *NIPS Workshop on Discrete Optimization in Machine Learning (DISCML)*, 2012.
- [12] T. Cover, “Geometrical and statistical properties of systems of linear inequalities with applications in pattern recognition,” *Electronic Computers, IEEE Transactions on*, no. 3, pp. 326–334, 1965.
- [13] F. Tanaka and S. Edwards, “Analytic theory of the ground state properties of a spin glass. i. ising spin glass,” *Journal of Physics F: Metal Physics*, vol. 10, p. 2769, 1980.
- [14] D. Gross and M. Mézard, “The simplest spin glass,” *Nuclear Physics B*, vol. 240, no. 4, pp. 431–452, 1984.
- [15] R. J. McEliece and E. C. Posner, “The number of stable points of an infinite-range spin glass,” *JPL Telecomm. and Data Acquisition Progress Report*, vol. 42-83, pp. 209–215, 1985.
- [16] M. Fulan, “A hopfield network with exponential storage capability,” *Master’s Thesis, Ohio State University*, 1988.
- [17] A. Salavati, K. Raj Kumar, A. Shokrollahi, and W. Gerstnery, “Neural pre-coding increases the pattern retrieval capacity of hopfield and bidirectional associative memories,” in *Information Theory Proceedings (ISIT), 2011 IEEE International Symposium on*. IEEE, 2011, pp. 850–854.
- [18] K. Kumar, A. Salavati, and A. Shokrollahi, “Exponential pattern retrieval capacity with non-binary associative memory,” in *Information Theory Workshop (ITW), 2011 IEEE*. IEEE, 2011, pp. 80–84.
- [19] E. Ising, “Beitrag zur Theorie des Ferromagnetismus,” *Zeitschrift für Physik*, vol. 31, pp. 253–258, Feb. 1925.
- [20] S. G. Brush, “History of the lenz-ising model,” *Rev. Mod. Phys.*, vol. 39, pp. 883–893, Oct 1967. [Online]. Available: <http://link.aps.org/doi/10.1103/RevModPhys.39.883>
- [21] S. Bernstein, “On a modification of chebyshevs inequality and of the error formula of laplace,” *Ann. Sci. Inst. Sav. Ukraine, Sect. Math*, vol. 1, no. 4, pp. 38–49, 1924.
- [22] G. Grimmett and D. Stirzaker, *Probability and Random Process*. Oxford University Press, 1992.
- [23] Y. Dekel, O. Gurel-Gurevich, and Y. Peres, “Finding hidden cliques in linear time with high probability,” *Arxiv preprint arXiv:1010.2997*, 2010.
- [24] S. Wicker and V. Bhargava, *Reed-Solomon codes and their applications*. Wiley-IEEE Press, 1999.
- [25] J. Hopfield and D. Tank, “Neural computation of decisions in optimization problems,” *Biological cybernetics*, vol. 52, no. 3, pp. 141–152, 1985.
- [26] P. Berkes, G. Orbán, M. Lengyel, and J. Fiser, “Spontaneous Cortical Activity Reveals Hallmarks of an Optimal Internal Model of the Environment,” *Science*, vol. 331, no. 6013, pp. 83–87, 2011.
- [27] H. von Helmholtz, *Handbuch der Physiologischen Optik*. Voss, Leipzig, 1867.
- [28] E. Schneidman, M. Berry, R. Segev, and W. Bialek, “Weak pairwise correlations imply strongly correlated network states in a neural population,” *Nature*, vol. 440, no. 7087, pp. 1007–1012, 2006.
- [29] J. Shlens, G. Field, J. Gauthier, M. Grivich, D. Petrusca, A. Sher, A. Litke, and E. Chichilnisky, “The structure of multi-neuron firing patterns in primate retina,” *Journal of Neuroscience*, vol. 26, no. 32, p. 8254, 2006.
- [30] J. Shlens, G. Field, J. Gauthier, M. Greschner, A. Sher, A. Litke, and E. Chichilnisky, “The structure of large-scale synchronized firing in primate retina,” *Journal of Neuroscience*, vol. 29, no. 15, p. 5022, 2009.
- [31] A. Barreiro, J. Gjorgjieva, F. Rieke, and E. Shea-Brown, “When are feedforward microcircuits well-modeled by maximum entropy methods?” *Arxiv preprint arXiv:1011.2797*, 2010.