

The Problem with Blondes

By Sara Robinson

One of the main complaints I hear from mathematicians about *A Beautiful Mind*, the movie based on the life of John Nash, is that there's lots of material about his mental illness but very little about his mathematical achievements. The only reference to any of his theorems is a bizarre bar scene in which Nash and several of his Princeton buddies are ogling a beautiful blonde woman who has just walked in, surrounded by a group of less attractive friends.

As the blonde poses and preens, the fictional Nash ponders the game theoretic problem before him. He reasons that if all the men went for the blonde, at most one would be successful and the rest would be forced to turn to her friends. Since the friends would be miffed at being second choice, those men would end up alone. Nash concludes that they would all do better by ignoring the blonde and going directly for her friends. That way, he reasons, no one would end up alone.

Crowing about his "discovery," the movie Nash skips past the bewildered blonde, inspired to work out the concept now known as Nash equilibrium, the basis of his Nobel Prize in Economics in 1994.

I didn't know what a Nash equilibrium was, but after seeing that scene I had to look it up. I found that a Nash equilibrium is a set of strategies (one for each player) expressed as probabilities that each of a number of choices will be selected, such that even if a player learns everyone else's strategy, he has no incentive to change his own.

In the well-known two-player game "Rock, Paper, Scissors," for example, a Nash equilibrium occurs when each player picks one of the three uniformly at random. It is outside the realm of two-person zero-sum games, however, that Nash's contribution becomes most interesting.

When this definition of a Nash equilibrium is applied to *A Beautiful Mind*, it becomes clear that director Ron Howard erred. The solution to the problem of the blonde, as depicted in the movie, is not a Nash equilibrium: If Nash (or any of his buddies) knew that all the others planned to go for the blonde's friends, he would do better by approaching the blonde. If Nash were to go for the blonde himself, however, while his friends went for her friends, this could be a Nash equilibrium, depending on the payoffs. But are there more Nash equilibria? And how should I, as a former mathematician, advise my friends to behave in bars?

To answer these questions and provide material for a sequel, I formally defined "The Problem with Blondes" as follows:

Suppose that each of n players is given a choice of either going for the blonde or doing nothing. Suppose also that doing nothing (the equivalent of going successfully for the friends) has a payoff of 2, and that going for the blonde successfully has a payoff of 3. Going for the blonde and losing out to some other bozo has a payoff of 0. Each player's goal, as in any social situation, is to maximize his expected payoff.

As for the blonde, suppose that she chooses her man by looking at all the men approaching her and picking one at random, possibly a reasonable approximation to the real-life behavior of blonde women. (OK, I confess: I'm a brunette.)

Question: What are all the Nash equilibria for this game?

I offer this puzzle to the *SIAM News* readership, along with a nice solution provided by my friend Leonard Schulman, a professor of computer science at Caltech, in the hope that it might be useful in social situations; Leonard's solution appears below.

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Solution: Describe a sequence of strategies for the players in the form of probabilities p_1, \dots, p_n that they will go for the blonde. If player i bids for the blonde, the probability that he will win her is:

$$W_i = W(p_1, \dots, p_{i-1}, p_{i+1}, \dots, p_n) = \sum_{k=1}^n 1/k P(k-1 \text{ of the players other than } i \text{ bid for the blonde}).$$

Lemma 1: W_i is a symmetric and strictly decreasing function of each of its arguments (i.e., for each $j \neq i$, as p_j increases from 0 to 1, W_i strictly decreases).

Notice that the expected payoff to player i is

$$(1 - p_i) 2 + 3 p_i W_i = 2 + p_i (3W_i - 2).$$

From this we get:

Lemma 2: In a Nash equilibrium, for every player i , either (a) $W_i > 2/3$ and $p_i = 1$, or (b) $W_i < 2/3$ and $p_i = 0$, or (c) $W_i = 2/3$.

Theorem: There are $2^n - 1$ Nash equilibria, each described by a nonempty subset S of the players. There are probabilities $1 = q_1 > q_2 > \dots > q_n$ such that the equilibrium point described by S , $|S| = k$, has

$$\begin{aligned} p_i &= q_k : i \in S \\ p_i &= 0 : i \notin S. \end{aligned}$$

Proof: Consider any equilibrium point. Notice that if for one player $p_i = 1$, then for all other players j , $W_j \leq 1/2$ and hence $p_j = 0$.

Otherwise, suppose that all $p_i < 1$. Then there must be at least two nonzero p_i : If p_i were uniquely greater than 0, then $W_i = 1$ and, by Lemma 2, $p_i = 1$.

All the nonzero p_i must in fact be equal, which can be shown by supposing to the contrary that $p_1 > p_2 > 0$. Then, by Lemma 1, $W(p_2, \dots, p_n) > W(p_1, p_3, \dots, p_n)$, so they cannot both be equal to $2/3$, contradicting Lemma 2.

All that remains is to note that for each nonempty subset of S , there is indeed a Nash equilibrium of the type described in the theorem, since $W(0, \dots, 0) = 1$, $W(1, \dots, 1) = 1/k < 2/3$, and since W is continuous. Q.E.D.

For the sake of completeness, let's see what q_k is.

For $k \geq 2$, q_k is the unique solution for $0 < q < 1$ of the equation

$$\frac{2kq}{3} = 1 - (1-q)^k.$$

Reason: For $k \geq 2$, consider a player who is in the set S . He, and every one of the other $k - 1$ players in S , bids for the blonde with probability q_k . Since $0 < q_k < 1$, he must fall into case (c) of Lemma 2. So $2/3 = W(q_k, \dots, q_k)$ where there are $k - 1$ arguments. Using the formula for W ,

$$2/3 = \sum_{i=1}^k \frac{1}{i} \binom{k-1}{i-1} q^{i-1} (1-q)^{k-i}.$$

Make the substitution $r = \frac{q}{1-q}$:

$$2/3 = \frac{(1-q)^k}{q} \sum_{i=1}^k \frac{1}{i} \binom{k-1}{i-1} r^i = \frac{(1-q)^k}{q} \sum_{j=0}^{k-1} \binom{k-1}{j} \int_0^r s^j ds = \frac{(1-q)^k}{q} \int_0^r (1+s)^{k-1} ds = \frac{1-(1-q)^k}{qk}.$$

For a given $k \geq 2$, this has a unique solution strictly between 0 and 1, since W is strictly decreasing in its arguments.

The first few q_k numerically are: $q_1 = 1$, $q_2 = 2/3$, $q_3 = (3 - 5^{1/2})/2 = 0.381966$, $q_4 = 0.266376$, $q_5 = 0.204324$, $q_6 = 0.165677$, $q_7 = 0.13931$, $q_8 = 0.120177$, $q_9 = 0.105662$, $q_{10} = 0.0942734$. Asymptotically, q_k is proportional to $1/k$.

You should decide for yourselves, but I plan to advise my friends to ditch the blonde and all of her friends and opt for a cold beer instead (payoff = 10). Many thanks to Leonard Schulman, though.