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Research

How the Gateway Arch Got its Shape

Abstract. Robert Osserman examines Eero Saarinen's Gateway Arch in St. Louis in order to shed light on what its exact shape is, why it is that shape, and whether the various decisions made during its design were based on aesthetic or structural considerations. Research included discussions with engineers and architects who worked with Saarinen on the project. The paper concludes by noting some questions that are still unanswered.

Introduction

Fig. 1. Eero Saarinen's Gateway Arch, St. Louis. Photo courtesy of Historic American Engineering Record, Library of Congress, Prints and Photograph Division

Much has been written on the subject of Eero Saarinen's most widely known creation and architectural landmark, the Gateway Arch in St. Louis. Nevertheless, it is difficult to find complete and authoritative answers to some of the most basic questions:

1. What is the shape of the Arch?
2. Why is it the shape that it is?
3. Of the various decisions that had to made, which were based on esthetic, and which on structural considerations?

These questions raise issues of a purely mathematical nature that are of independent interest. A number of them have been treated elsewhere [Osserman 2010] and will be referred to as needed. Here we focus on the basic questions that are our central concern.

The Shape of the Arch

The terms most often used in describing the shape of the Gateway Arch are *parabola, catenary,* and *weighted catenary.* We shall discuss each of those terms in some detail later on, but let us note that they all represent mathematical idealizations of physical phenomena. A parabola, as Galileo demonstrated, is the shape of a path traversed by a projectile subjected only to its initial impetus together with the force of gravity, in the absence of air resistance. A catenary is the shape assumed by a hanging chain or a flexible cord of uniform density. A weighted catenary is the shape assumed by a hanging chain whose links vary in size or weight, or by a flexible cord of variable width, or variable density material.[1]

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An equally basic feature of the shape of the Arch is what we shall call, borrowing a term from computer, television, and movie screens, its "aspect ratio": that is, the ratio of width to height of the inside dimensions of the smallest picture frame that can hold the full frontal view of the Arch; or, in architectural drawing terms, a front elevation of the Arch. Aspect ratios are usually expressed in a form such as "3-to-2" or "3:2."

An occasional source of confusion stems from the fact that both the parabola and the catenary extend to infinity. In both cases there is, up to scaling, a single curve, but arcs of a single parabola or catenary can look very different from each other, depending on the part that one chooses, and the corresponding aspect ratio of the given arc. In fact, in both cases, one may choose arcs with any aspect ratio one wishes. Fig. 2 illustrates how to obtain a variety of aspect ratios from a single catenary.

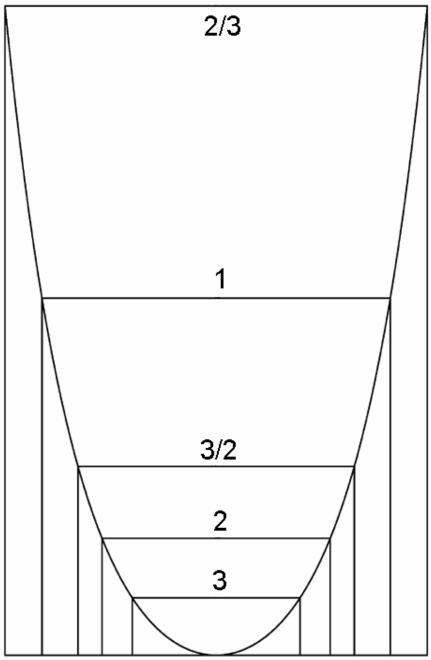

Fig. 2. Aspect ratios for a catenary

A further potential source of confusion is that Saarinen changed the aspect ratio of his Arch design between the time that he originally won the competition, and the final design used in the construction. The original design was for an arch that was 630 feet wide and 590 feet high, whose aspect ratio was therefore approximately 21:20, or just shy of square. By the time the arch was built, Saarinen had kept the original width, but raised the height to 630 feet, exactly matching its width. According to architect Bruce Detmers,[2] who worked with Saarinen on the Arch, it was typical of Saarinen to favor simple geometric figures, such as a square, and therefore not surprising that he would modify his design to obtain the 1:1 aspect ratio. According to other sources, the principal reason that Saarinen increased the height of the arch was that between the time that he won the original competition in 1948, and the beginning of construction in the early 1960s, new and higher buildings had been added to the surrounding landscape, which forced Saarinen to modify his own plans so that the Arch would clearly dominate its surroundings. It is undoubtedly the case that both of these factors played a role in his final choice of the exact 1:1 aspect ratio.

It may be worth noting that the 1:1 aspect ratio is a very old tradition in architecture. In medieval times it was known as *ad quadratum.* The original 1386 specifications for the cathedral of Milan is an example (cf. [Heyman 1999: 19]).

The contrast between the shape of a parabola and a catenary is clear if we choose segments of each with a 1:1 aspect ratio (fig. 3).

In fig. 3, the parabola is the pointier inner curve and the catenary the rounder outer curve.

Fig. 3. Parabola and Catenary with aspect ratio 1:1

History of the problem

Commemorative arches designed to be lasting monuments date back thousands of years, with a number still standing from the days of ancient Rome. However, it does not appear to have been until 1675 that the question was raised – at least in print – of what shape an arch *should be,* from a mathematical and structural point of view. At that time, one of the leading scientists of the day,[3] Robert Hooke, provided his answer to the question in the form of an anagram of the Latin phrase: *ut pendet continuum flexile, sic stabit contiguum rigidum inversum,* or, "as hangs the flexible line, so but inverted will stand the rigid arch." Hooke was famous for his ability to devise physical demonstrations and experiments illustrating both old and newly discovered scientific principles, and was specifically commissioned to do that for the recently created Royal Society. In the case of the ideal shape of an arch, he realized that what one wants is that the slope of the arch at each point exactly match the combined horizontal and vertical forces acting on that part of the arch – the vertical component being due to gravity from the weight of the portion of the arch lying above the point, and the horizontal force being simply transmitted unchanged along the arch. At the apex of the arch, there is no weight above it, hence no vertical component, and the force is simply the horizontal one of the two sides of the arch leaning against each other. As a result, the slope at that point is zero. As one moves down along the arch, the vertical force keeps increasing, and the slope of the arch must increase accordingly.

One simple consequence of this general reasoning is that at the base of the arch, the horizontal force continues to be present – a fact well known to builders who created flying buttresses and other devices to counter that force – and as a consequence, the bottom of the arch should never be strictly vertical, but rather, angled outwards. As it happens, probably not by chance, Hooke recorded his dictum at the time when both he and Christopher Wren were among the chief architects in charge of surveying the damage, and of rebuilding London after the disastrous fire of 1666. Hooke shared his insight with Wren, who immediately applied it to his design of St. Paul's cathedral, which features an interior dome that is the first to be angled out at its base, rather than vertical (cf. [Heyman 1999: 40-41]).

But the main thrust of Hooke's observation was that the exact shape of an ideal arch could be obtained by simply hanging a chain (or "flexible line") and recording the form that it takes. The equilibrium position would be determined by a balance of vertical and horizontal forces that are the mirror image of what they would be in the case of the arch, with the role of gravity reversed: the vertical component at each point is determined by the weight of the chain below it that needs to be supported, while the horizontal component is simply transmitted unchanged along the chain.

Left unanswered by Hooke is the obvious mathematical question raised by his purely physical solution to the ideal shape of an arch: "what is the equation of the curve that is formed by a hanging chain?" Wherever this subject is mentioned in print, one is likely to find a statement to the effect that Galileo was the first to raise the question, and that he answered incorrectly that the shape that the chain would take was a parabola. Both of those statements merit a closer look.

Galileo's discussion of the shape of a hanging chain appears in print in his last, and in the opinion of many, his best, book, *Two New Sciences*. The book takes the form of four days of dialogue on a wide array of subjects. The first day introduces the first "new science": the analysis of the strength of beams and columns from a scientific and mathematical point of view. It is often considered as the incubator of the modern fields of structural engineering and strength of materials. During the second day, Galileo has his chief protagonist, Salviati, propose two practical methods of drawing a parabola [Galilei 1974: 143]. The first is by rolling a ball along a tilted metal mirror, and the second is by hanging a fine chain, and marking points along it. In other words, he does not pose the question of the shape taken by a chain, he simply assumes that it takes the form of a parabola, based on both theoretical reasoning, using the decomposition of the forces acting on it in the vertical and horizontal directions that led him to deduce the parabolic path of a projectile, as well as on experimental evidence: drawing a parabola on a vertical board, and observing the shape of a hanging chain that seems to follow the same curve.

Few commentators note that Galileo returns to the same question on the fourth day of his dialog, and explicitly states that the similarity is only approximate. He also explains the analogy between the forces acting on a projectile, and those on a hanging chain. Here is the exact quotation:

> Salviati: The curvature of the line of the horizontal projectile seems to derive from two forces, of which one … drives it horizontally, while the other … draws it straight down. In drawing the rope, there is [likewise] the force of that which pulls it horizontally, and also that of the weight of the rope itself, which naturally inclines it downward. So these two kinds of events are very similar.
>
> …
>
> But I wish to cause you wonder and delight together by telling you that the cord thus hung, whether much or little stretched, bends in a line that is very close to parabolic. The similarity is so great that if you draw a parabolic line in a vertical plane … and then hang a little chain from the extremities … , you will see by slackening the little chain now more and now less, that it curves and adapts itself to the parabola; and the agreement will be the closer, the less curved and the more extended the parabola drawn shall be. In parabolas described with an elevation of less than 45°, the chain will go almost exactly along the parabola [Galilei 1974: 256-7].

In short, Galileo never poses the question of precisely what shape is taken by the hanging chain, and he contents himself with noting that it provides a close approximation to a parabola, especially when the parabolic arc that one draws is near to the vertex where the shape is relatively flat. For example, a 45° parabola would be represented by the equation

$$y = \tfrac{1}{2} x^2, \quad -1 \leq x \leq 1,$$

which fits in a rectangle of width 2, and height ½, so that its aspect ratio is 4:1. Fig. 4 shows the parabola together with a catenary having the same aspect ratio.

Fig. 4. 45° parabola and catenary with the same aspect ratio: 4:1

It is often pointed out that unlike the case of a freely hanging cable which will take the form of a catenary, when a cable supports a roadway that is much heavier than itself and whose weight is distributed evenly along the horizontal rather than equally along the cable, the cable will take the shape of a parabola. That will be the case for the cables joining the two towers of a suspension bridge.

Fig. 5. The Golden Gate Bridge

In the case of the Golden Gate Bridge, shown in fig. 5, the central cables form curves that make an angle of almost exactly 30° with the horizontal at the points where they leave the towers. However, as fig. 6 makes clear (or as one can show by a direct calculation), the parabola and approximating catenary are virtually indistinguishable in this case.

Fig. 6. 30° parabola and corresponding catenary

The fact that the overwhelming number of references to Galileo's treatment of this subject declare him to have been in error, and ignore the fact that he ends up getting it exactly right, would seem to reflect an extra degree of pleasure in catching him out in any perceived slip-up. In fact, the first page to come up on a Google search of "Galileo wrong" starts with an item from NASA entitled "Was Galileo wrong?" and ends with a NOVA television program entitled "Galileo's battle for the heavens: his big mistake." The first of these entries questions whether the acceleration of gravity is the same for all varieties of materials – a fundamental tenet of science ever since Galileo, and certainly true within the limits of measurements available to him (and many subsequent generations). The last has to do with an actual mistake: his theory that tides were caused by the rotation of the earth. In between these two items are a series of references to a book entitled "Galileo was Wrong: The Church was Right," devoted to proving that the Earth in fact *is* motionless and at the center of the Universe, just as Galileo's opponents asserted at the time.[4]

Of course we now know that had Galileo posed the question of finding the exact shape of a hanging chain, he could not have answered it, since the basic tool needed had not yet been invented: the calculus. It was not until the end of the seventeenth century that the problem was explicitly formulated and solved, the curve in question being dubbed a "catenary" since it was formed by a chain – a *catena*.[5] The brilliant but cantankerous Bernoulli brothers, Jacob and Johann, produced a series of derivations of the equation for the catenary along with a series of barbs at each other's reasoning, such as Jacob's parody of an argument of Johann that he describes as somewhat like proving that a pebble is stone by reasoning that "every man is stone; every pebble is a man; therefore every pebble is stone."[6] Whatever the shortcomings of the reasoning, they did indeed arrive at the correct conclusion. The equation for the catenary can be written in the form we now call the hyperbolic cosine,

$$y = \cosh x = \tfrac{1}{2}\,(e^{x} + e^{-x})$$

in suitable choice of coordinates.[7]

The design of the Arch

The Gateway Arch was only one component – although clearly the most dramatic one – of Eero Saarinen's winning entry in the 1947 open competition for the design of a "Jefferson National Expansion Memorial," dedicated to Thomas Jefferson – to his vision of an America stretching across the continent, and his Louisiana Purchase, which roughly doubled the size of America at the time. By a truly strange coincidence, the first use of "catenary" recorded in the Oxford English Dictionary is by the future President Thomas Jefferson, in a letter dated December 23, 1788 to Thomas Paine, recommending the use of a catenary arch rather than a circular one for a 400-foot span iron bridge that Paine is proposing to build. Jefferson would certainly have been inordinately pleased, as an architect himself, to know that the principle of the catenary would form the basis of what was to be the largest monument in America in the twentieth century, in honor of Jefferson himself.

The Jefferson Memorial competition attracted virtually all the major architects in America, including Eero's father Eliel, a much more famous architect at the time than his 37-year old son. At least one part of the confusion involving the shape of the Gateway Arch stems from the fact that the competition was held in two parts, first narrowing down to the top five contenders, and then – after significant revisions were made – choosing the winner among those five. In addition, there was a gap of 15 years between the time that Saarinen's proposal was chosen and when construction actually began, during which time still further changes were adopted. And even after construction had started, alterations were made, as one can see from different sets of blueprints in the Saarinen archives at Yale University. Further confusion arises from a certain degree of carelessness in terminology that can be seen, for example, in a dictionary of architecture where a picture of an arch is accompanied by a caption referring to it as a "catenary or parabolic arch" [*Visual Dictionary of Architecture* 2008: 41], not aware of, or not interested in, fine distinctions. Whatever the reason, newspapers and architectural journals that reported or commented on the shape of Saarinen's winning design for the Gateway Arch, referred to it without exception as a parabola.[8] By chance, among the letters that Saarinen received was one from H. E. Grant, head of the Department of Engineering Drawing at Washington University in St. Louis, asking for more details about the "parabolic arch," since Grant wanted to use it as an example in a book on descriptive geometry that he was writing at the time. In his response, dated March 24, 1948, Saarinen says:

> The arch actually is not a true parabola, nor is it a catenary curve. We worked at first with the mathematical shapes, but finally adjusted it according to the eye. I suspect, however, that a catenary curve with links of the chain graded at the same proportion as the arch thins out would come very close to the lines upon which we settled."[9]

In other documents Saarinen left considerable room for ambiguity. In 1959 he wrote:

> The arch is not a true parabola, as is often stated. Instead it is a catenary curve – the curve of a hanging chain – a curve in which the forces of thrust are continuously kept within the center of the legs of the arch.[10]

Whether by "catenary curve" he means an actual catenary, or is trying to hedge a bit, is not clear. In an unpublished transcript dating from 1958-59, he describes the Arch as "an absolutely pure shape where the compression line goes right through the center line of the structure directly to the ground. In other words, a perfect catenary" [Eero Saarinen 2006: 343].

If one consults the Wikipedia entry for "catenary" one finds – at least at the time of this writing (June 2009):

> The Gateway Arch in Saint Louis … follows the form of an inverted catenary. … The exact formula
> $$y = -127.7 \text{ ft} \cdot \cosh(x/127.7 \text{ ft}) + 757.7 \text{ ft}$$
> is displayed inside the arch.

The equation given is indeed that of a catenary, and it has the desired dimensions of 630 feet in both height and width, but it is not the equation that was used in the construction of the arch, and that appears on all the blueprints and specifications.[11]

At the Arch itself, a sheet provided by the National Park Service, under whose jurisdiction the monument falls, is headed "Equation for the catenary curve of the centroid of arch cross-section." Below that is the same equation that appears on the blueprints. We shall return to the equation shortly, but it is definitely not a catenary.

In 1983, an article entitled "Is It a Catenary? New questions about the shape of Saarinen's St. Louis Arch," appeared in an architectural journal [Crosbie 1983]. The author, Michael J. Crosbie, explained that despite widespread belief that the Gateway Arch is shaped like a catenary, it is not. Instead, it is a "weighted catenary," the form taken by a "weighted chain" in which the various links are of different weight, rather than all the same. In mathematical terms, it amounts to using a "flexible line" with a density that is variable rather than constant. It appears to be uniformly true that any description of the shape of the Gateway Arch that tries to be accurate will use the term "weighted catenary" or something similar.

We are now in a position to state precisely the mathematical questions that arise.

1. How much information is conveyed by the expression "weighted catenary"? More precisely, what range of curves may be obtained by suitable choice of a variable density?
2. What was the basis for the choice of the particular "weighted catenary" chosen by Saarinen? Was it primarily on esthetic or on structural/mathematical grounds?
3. The arch is designed so that its cross-sections are equilateral triangles. Again the question whether that choice was esthetic or structural.
4. The size of the cross-sections varies according to a precise formula, with the sides of the equilateral triangles growing from 17 feet at the apex to 54 feet at ground level. What determined those dimensions, and what is the significance of the specific formula used for the tapering? And once again, were those choices motivated by esthetic or mathematical considerations?

The answer to the first question can be formulated quite simply: the term "weighted catenary" conveys essentially no useful information; virtually any curve that one can picture as a possible candidate[12] for a weighted catenary can actually be reproduced by a suitably weighted chain.[13] Among those are, for example, a parabola and a circular arc, as well as the catenary-like curve of the actual arch. In all of those cases one can explicitly say how to distribute the weight so that the chain, or flexible line, will hang in exactly the desired shape. In other words, saying that the shape of the Gateway Arch is a weighted catenary is technically correct, but essentially devoid of content. Furthermore, it is not descriptive, in that it says nothing about the actual shape of the Arch, but only about the method used to produce it.

On the other hand, there is a very simple description of the precise shape of the Arch. It is what we call a "flattened catenary" and consists simply of a catenary that has been shrunk uniformly in the vertical direction by a given amount. It will have a vertical axis of symmetry, just as the catenary does, and if we choose that axis to be the y-axis, then the equation of a flattened catenary takes the form

$$y = A \cosh Bx + C, \qquad (1)$$

where $A,B > 0$. Note that if we take the catenary curve $y = \cosh x$, and scale it uniformly up or down in size, we get the equation $By = \cosh Bx$, while flattening it by a uniform compression in the vertical direction would give the equation $y = D \cosh x$, where

$$0 < D < 1.$$

In other words, equation (1) represents a catenary if and only if $A = 1/B$. The constant C corresponds simply to a translation in the vertical direction, and we if choose coordinates so that the vertex of the curve is at the origin, then $C = - A$. Putting all this together, we see that by setting $D = AB$, equation (1) takes the form

$$y = D(1/B)(\cosh Bx - 1) \qquad (2)$$

so that equation (2) represents a catenary with vertex at the origin that has been flattened vertically by the factor D.

The centroid curve of the Gateway Arch is of exactly this form, where A and B are specified numerical constants:

$$A = 68.7672, \;\; B = .0100333, \qquad (3)$$

so that the flattening factor D is given by

$$D = .69 \qquad (4)$$

In other words, the catenary is shrunk in the vertical direction by just under a third. The effect is to "round it out" somewhat more at the vertex. The y-coordinate in equation (2) for the Gateway Arch represents the vertical distance *down* from the vertex of the centroid curve. The numerical values given in (3) correspond to distances measured in feet.

It is worth noting that shrinking a curve in the vertical direction is exactly equivalent, up to uniform scaling, to expanding it uniformly in the horizontal direction – precisely what is often done to reformat a film done with one aspect ratio to fit it onto a wider screen with a different aspect ratio (fig. 7).

It is also important to observe that since the flattening degree D for the Arch given by (4) is a little over 2/3, and since the centroid curve of the Arch is intended to have an aspect ratio of approximately 1:1, ("approximate" because it is the outer silhouette of the Arch that is designed to have aspect ratio 1:1, and the same will not be true of the centroid curve, as we discuss more fully below,) we must start with a portion of the catenary having an aspect ratio of approximately 2:3 and either shrink it vertically by about 2/3, or stretch it horizontally by about 3/2 (fig. 8).

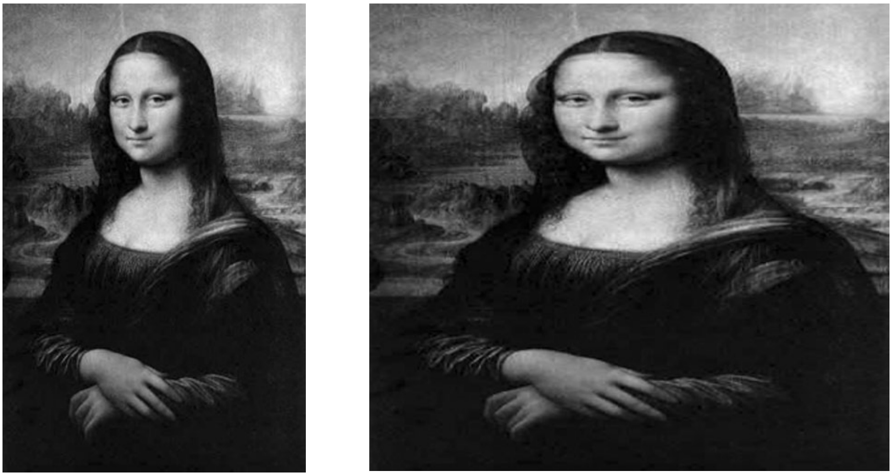

Fig. 7. Wide-screen Mona Lisa

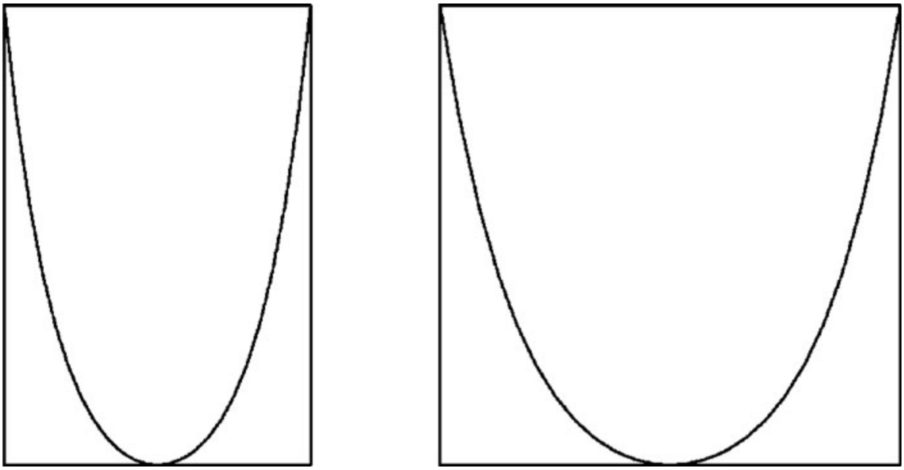

Fig. 8. Re-formatting to obtain a 1:1 aspect ratio

It is ironic that Saarinen has been attacked from both sides, with some commentators criticizing the shape of the arch for being too purely mathematical to be taken seriously as a work of art, while others take him to task for not being "pure" enough. Michael Crosbie [1983] notes the fact that "the curve is not a perfect catenary," adding "— a pure form that would seem more attractive to Saarinen…". Hélène Lipstadt is more severe: "The purity is illusory and the explanation deceptive (if not deceitful), since he knowingly had his engineers calculate an impure, weighted catenary" (quoted in [Merkel 2005: 244, note 36]).

Clearly, a catenary is no higher on the scale of purity than, say, a parabola or a circular arc, which are "weighted catenaries." The equation for the parabola is perhaps the simplest of all equations for a curve, consisting as it does of a polynomial with a single (quadratic) term. A circular arc is one step more complicated, involving square roots, and hence, algebraic functions. The catenary, built out of exponential functions, which are not even algebraic, but so-called transcendental functions, would seem to lie relatively far afield in the domain of mathematical purity. Presumably what is meant by the criticism is that the equation Saarinen used as a basis for his arch, on the suggestion of Hannskarl Bandel, one of his engineers, suffers from being *almost*, but not quite, a catenary, involving the insertion of rather arbitrary-looking numerical constants.

To criticize Saarinen for not using a perfect catenary, however, is to miss the point of Hooke's original insight. The catenary is the shape one obtains using a uniform chain or "line" and is therefore the shape one wants for an arch of uniform thickness. For a relatively small arch, uniform thickness would not be unreasonable, but for the kind of monumental structure proposed by Saarinen for his Arch, it would be almost unthinkable, both on esthetic and structural grounds. One would naturally want the cross-section of the arch to be the least near the apex, where it has little or nothing to support and where any additional weight adds to the strength requirements of everything below. Conversely, the parts of the arch closest to the ground have the most weight to support, and would naturally have the largest cross-sectional size. As a consequence, the correct shape would not be the mirror image of a uniform chain, but of a correspondingly weighted chain, as Saarinen points out in the quotation given above. That principle was clearly understood early on. A famous illustration from 1748 in a report by Giovanni Poleni on the cracks in the dome of St. Peter's shows a cross-section of the dome, together with the shape of a hanging chain that was weighted proportionally to the loads of the corresponding sections. Poleni used the result to deduce that the structure was basically sound, despite the cracks that had appeared [Heyman 1999: 38-39].

In short, a "pure catenary" is the ideal shape for an arch of uniform thickness, and a flattened catenary is the ideal shape for an arch that is tapered in a certain precise (and elementary) manner. In the case of a weighted chain, we may express the amount of weighting by a density function $\rho(s)$, where the weight of any arc of the chain is the integral of $\rho(s)$ with respect to arclength s over that arc. One can show (see [Osserman 2010] that a given flattened catenary of the form $y = f(x) = A \cosh Bx + C$ is obtained from a density function ρ which, when expressed in terms of x, takes the form

$$\rho(s(x)) = (ay + b)/(ds/dx) = (a f(x) + b)/\sqrt{(1 + (f'(x))^2)},$$

where the coefficients a and b are determined by the coefficients A, B, and C of $f(x)$, up to an arbitrary multiple (since multiplying the density by an arbitrary positive constant just changes the total weight, but not the shape of a hanging chain.) What this means physically is that the weight of any portion of the chain lying over a segment of the x-axis, given by $\int \rho(s) ds$, is equal to $\int (ay+b) dx$ over the given x-interval.

That brings us to the answer to our fourth question: how the shape of the Arch – which is to say, the shape of its centroid curve – is related to the degree of tapering.

As we have noted, the centroid curve of the Gateway Arch is a flattened catenary whose equation is of the form (1), where the x-axis is horizontal, the origin is at the vertex, or highest point of the curve, and y represents the distance down from the vertex. We denote by h the height of the curve, and by w its width. The fact that y=0 when x=0 translates to the condition $C = -A$, so that the equation takes the form

$$y = A (\cosh Bx - 1). \qquad (5)$$

Then the fact that $y = h$ when $x = \pm w/2$ implies

$$h = A(\cosh Bw/2 - 1). \tag{6}$$

If we introduce the notation

$$R = \cosh Bw/2 \ ,$$

then equation (6) becomes

$$A = h/(R - 1) \tag{7}$$

while from the definition of R,

$$B = (2/w) \cosh^{-1} R \ . \tag{8}$$

The coefficients A, B that enter into the equation for the centroid curve given on the blueprints and all the official documents for the Gateway Arch have exactly this form (fig. 9).

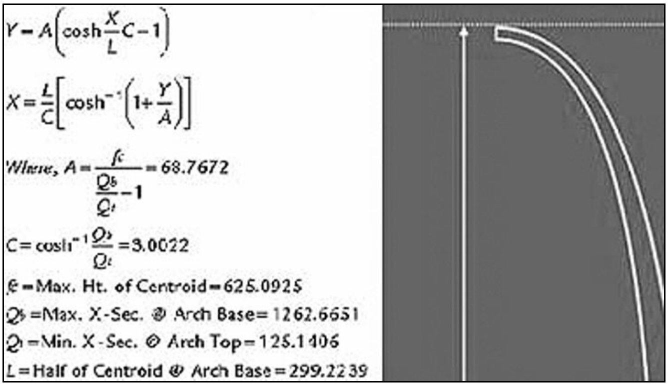

Fig. 9. Official equations for the Arch.
Source: http://www.nps.gov/jeff/planyourvisit/mathematical-equation.htm

We see that their somewhat strange appearance arises simply from the condition that the curve pass through the points (0,0) and (w/2, h). The only question is the value of the constant R. By its definition, we have $R > 1$, and conversely, if we choose an arbitrary number $R > 1$, then equation (5) with coefficients defined by (7) and (8) represents a flattened catenary passing through the given points. We find further that the degree of flattening D of the curve is given by

$$D = AB = (h/w)(2 \cosh^{-1} R) /(R-1) \tag{9}$$

Or, equivalently,

$$D = (1/r)(2 \cosh^{-1} R) /(R-1) \tag{10}$$

where we have denoted by r the aspect ratio of the centroid curve,

$$r = w/h \ . \tag{11}$$

In other words, the flattening coefficient D is completely determined by the desired aspect ratio r and the – so far – arbitrary constant R.

We are finally in a position to spell out precisely the relation between the shape of the Arch and the basic geometric decisions that were made. In particular we see that the complicated-looking values of the coefficients A and B in the equation of the Arch are direct consequences of a few simple choices. Those choices are:

(i) the centroid curve be a flattened catenary;

(ii) the orthogonal sections be equilateral triangles with one vertex pointing inward;

(iii) the triangles at the top and the bottom be 17 and 54 feet on a side respectively;

(iv) the overall height and width each be 630 feet;

(v) the constant R in equations (7) and (8) be given by $R = Q_b / Q_v$, where Q_b is the cross-sectional area of the triangle at the base, and Q_v the corresponding area at the top.

From (ii) and (iii) it follows that the cross-sectional areas at the vertex and the base are

$$Q_v = 125.1406, \quad Q_b = 1262.6651, \qquad (12)$$

so that according to (v), the value of R is a little over 10. Also from (ii) and (iii) we can calculate the height h and width w of the centroid curve needed to make the outside dimensions of the arch satisfy (iv), and they in turn determine the aspect ratio r. Inserting all these quantities into the equations (7), (8), and (9), gives the numerical values of the coefficients A and B that we noted earlier, as well as the degree D of flattening of the curve.

The one remaining choice that must be made is a formula for the tapering from top to bottom of the Arch. The decision there was to make the cross-sectional area $Q(x,y)$ at the point (x,y) of the centroid curve a linear function of y, interpolating between the given values (12) at the top and bottom. In other words,

(vi) $Q(x,y) = Q_v + (Q_b - Q_v)\, y/h$.

Conditions (i) – (vi) together provide a complete mathematical description of the basic design of the Gateway Arch. Its actual construction, with an outside skin consisting of flat stainless steel plates, requires dividing this basic shape into sections,[14] each bounded by a pair of equilateral triangles, and replacing the curved surface defined by the equations by three flat pieces.[15] The result is also completely defined mathematically, but is an artifact of construction, intended to so closely approximate the curved surface that one is not even aware, for example, that the inner curve of the Arch, its intrados, is not a smooth curve, but a polygonal one. That is indeed the case, and we limit our discussion here to the curved surface of the Arch that underlies the construction.

What remains to answer is whether the choice of the shape itself was dictated largely by structural or by esthetic considerations. When this question was put to Bruce Detmers, who worked with Saarinen on the Arch, his answer was unequivocal: Saarinen's choice was purely esthetic. According to Detmers, Saarinen worked with a large variety of shapes, many of them made on the elaborated Hooke principle, by hanging a weighted chain, or a piece of rubber cut out with a varying width to gain the same effect. The resulting shapes were then used, upside down, to make sizeable models of the arch, and Saarinen would examine the models from all angles. He ended up rejecting them all, as either too flat or too pointed, or otherwise unappealing. He then turned to the engineer Hannskarl Bandel for further ideas, and it was Bandel who suggested the shape that Saarinen found to his liking and that was the one actually used for the design of the Arch. (As a parenthetical note, the key role of the engineers throughout the process, from design to completion, is all-too-often forgotten in accounts of the Arch and its construction.)

The question that arises, if one accepts that Saarinen chose the shape of the Arch on purely esthetic grounds, is "What role, if any, was played by mathematics?"

Bruce Detmers notes two such roles. First, mathematics was critical as a means of communication. It is one thing to draw a shape and to make models that are 10 or 12 feet high, but if one arrives at a pleasing design, then it is still necessary to convert that design into specific instructions to provide those who must manufacture and assemble the parts. There is no obvious way to make the transition from a sketch or a model that must be scaled up by several orders of magnitude and described with absolute precision in order to arrive at the full-sized structure. But as soon as the shape is defined by a mathematical equation, one may build it on any scale that the available materials and methods will allow.

Second, Detmers also alluded to Saarinen's predilection for perfect geometric shapes like a square or a circle. In the case of the Arch, as we have already noted, he decided that he would like its proportions to be such that it could be exactly inscribed in a square.

An additional, and crucial role of mathematics was to calculate the ideal shape of the fully three-dimensional structure whose central curve had the form decided on. That comes down to a process that is the reverse of starting with a hanging chain and adopting the form it takes in order to build an arch. Here one starts with a desirable shape, and has to determine the degree of tapering that would be structurally ideal to go with that shape. But as we have noted, to virtually any shape defined by a mathematical equation, one can assign a weighting that will produce exactly that shape.

The history of the particular equation suggested by Bandel for the Arch goes back to the mid-nineteenth century. It arose in trying to determine the ideal shape for an arch that supports an earth bridge – a bridge formed by simply piling up dirt above the supporting arch to make a flat road surface. The steepness of the arch at each point should be determined by the balance of the horizontal and vertical forces at the point; the horizontal force is just transmitted along the arch and is unchanged from point to point, while the vertical force results from the weight of the column of dirt directly above the point, and is therefore proportional to the height of the road above that point on the arch. From that one can show that the ideal shape must be a flattened catenary (see, for example [Heyman 1982: 48-49]).

In the case of the Gateway Arch, it does not support any other structure. However, the effect of the tapering is that the weight being supported by each cross-section increases steadily as one moves from the top to the bottom of the arch, just as it does for an earth bridge. What emerges then is a simple answer to our second and fourth questions regarding the shape of the Arch and the amount of tapering: the two are inextricably entwined, as are the esthetic and structural decisions that have to be made.

The reality is that far more comes into play than these basic considerations. In any project of the magnitude of the Gateway Arch, economic and political factors are inevitably going to be involved. One striking example is the appearance of the George

Washington Bridge, with its two silvery towers of interlaced steel girders. According to the original design, those towers were to be covered in stone in the manner of the bridge's famous forerunner, the Brooklyn Bridge, and the steel framework one now sees were to be the invisible support. However, the money for the project ran out, and generations of New Yorkers, along with millions of out-of-town visitors, have engraved in their minds a very different-looking bridge than the one envisaged by its designers. The Eads Bridge across the Mississippi that now forms a backdrop to the Gateway Arch was as dramatic for its time as the George Washington Bridge across the Hudson several generations later. In both cases, the length of the bridges and the size of the river spanned broke new ground, and in both cases the bridges linked two adjacent states. In the case of the Eads Bridge, connecting Missouri to Illinois, powerful political factors entered in, affecting the final design of the bridge.[16]

Even on the structural level, many factors must be considered beyond the ideal shape for supporting the weight of the Arch itself. There are many forces besides gravity that must be taken into account, including those arising from potential gale-force winds, from earthquakes, and from uneven expansion and contraction as the sun heats up certain parts more than others. In addition, one must assure that the Arch is not susceptible to vibrations at certain resonant frequencies that might lead to structural failure, as happened in the famous case of the Tacoma Narrows Bridge.

Finally, there are many additional decisions that must be made, where esthetic and structural decisions are intermingled. For the overall effect, the most important was undoubtedly the choice of stainless steel for the outer surface, reflecting blue skies and clouds during the day, shining brilliantly against a dark sky when lit up at night. And right down to important details, such as the choice of whether to "brush" the steel horizontally or vertically, and what method to use for attaching adjoining steel plates.

That leaves us with one remaining question to be answered – question 3 about the choice of a triangular cross-section. The answer forms part of the subject of our final section.

Some controversial aspects of the Gateway Arch

No project as monumental as the construction of the Gateway Arch is apt to come to completion without arousing serious controversy. At the least, there are always matters of taste and a ready chorus of critics whose job it is to criticize. In the particular case of the Arch, there were four flags raised against Saarinen that are worthy of note.

The first was raised at the very outset, during the original competition: "What good is it?" A great deal of (public) money is to be spent on a structure with no "purpose" other than merely symbolic. (That is a criticism familiar to most mathematicians who engage in and teach "pure" mathematics, often with no practical applications in sight.)

It should be said that the Gateway Arch was designed with the express purpose of putting St. Louis back on the map after a long period of decline from its nineteenth-century glory as the second largest port in America and the "Gateway to the West." In that capacity, it has been wildly successful, attracting millions of visitors each year, and joining the Washington Monument and the Eiffel tower as among the most widely known symbols of their kind.

The unqualified success of the Arch, both symbolic and practical, as a draw of both attention and visitors to St. Louis, has pretty well answered this objection. Beyond that, many are struck by the beauty of the shape, and the issue then turns on the value of a piece of art or architecture that is admired and gives pleasure of a purely esthetic kind.

The second controversy arose during the construction of the Arch, when serious worries were expressed about its potential to fail and to collapse in what would have been a major catastrophe. That conflict and its resolution have been beautifully described by one of the principal figures involved, George Hartzog.[17]

The third regards our question about the choice of a triangular cross-section. Saarinen's original design for the Arch had a quadrilateral cross-section. Between the first and second phase of the competition, one of his colleagues, the sculptor Carl Milles, suggested that a triangular cross-section would be more attractive. Saarinen liked the idea and adopted it in his final designs, but he failed to publicly credit Milles with the suggestion, a failure that caused many hard feelings and much anger.[18]

The fourth controversy, and most interesting from many points of view, arose shortly after the proposal submitted by Saarinen and his team of co-workers was awarded the commission to build the monument. An article in the *New York Herald Tribune* displayed side-by-side pictures of the design submitted by Saarinen and a poster for an International Exposition that was planned for 1942 in Rome. The headline was "St. Louis Arch For Memorial Called Fascist." In fact, the Rome exposition was explicitly billed as commemorating the twentieth anniversary of Fascism in Italy, and Mussolini's "March on Rome" in 1922. The poster features an arch designed by the Italian architect Adalberto Libera that appears to be on much the same scale as that envisaged by Saarinen, and seems to resemble it strongly in shape. Saarinen vehemently denied knowing anything about the earlier proposed arch, which in fact was never built, due at least in part to the advent of World War II. He further noted that a "simple form, based on the natural laws of mathematics," could no more be co-opted by a political movement than the Washington Monument could be faulted for the fact that its design is that of ancient Egyptian obelisks that were built by slaves.

Ironically, Saarinen's defense – that the form of his Arch was a standard one that could well have been used by anybody – has the effect of diminishing the originality of his design. On the other hand, it provides fairly convincing evidence that he knew nothing about the earlier design. The resemblance on the poster turns out to be purely an artifact of the particular angle at which the Libera Arch is depicted. The actual plans submitted by Libera show that his proposal was for a strictly semi-circular arch, with no resemblance to Saarinen's, other than the overall similar scale, and of course the fact that both arrived at the same idea of a monumental arch as a symbolic gateway – in Libera's case, to the proposed Exposition and the associated newly rebuilt area of Rome, still known as EUR: Esposizione Universale Roma – and for Saarinen, to the broad Western expanse of America across the Mississippi River.

An additional irony is that when Saarinen's design acquired renown after he won the competition, Libera accused him of plagiarism, whereas Libera had himself been similarly charged, because another team of Italian architects had suggested much the same design for an entry arch to the EUR somewhat earlier than Libera.

The final irony is that even if World War II had not intervened, it is not clear that Libera's arch would have been built, or that if built, it would have been able to resist collapse. In fact, a semi-circle is far from an ideal shape for a monumental arch. Serious doubts were expressed at the time. Some of those doubts concerned the choice of materials and structural details of Libera's proposal. Others are intimately connected with Robert Hooke's original insight connecting the shape of an arch with that of a hanging chain. As we noted earlier, one consequence, put to use immediately by Christopher Wren, is that one wants the base of the arch to hit the ground at some angle, and not vertically, as is the case of a semi-circle. The larger the size of the arch, the more vital such considerations become. That is true for an arch of uniform thickness and the corresponding chain of uniform weight. But it is equally true for a weighted chain of any sort.

We have also noted earlier that although it may seem completely counter-intuitive, it is possible to weight a chain in such a manner that it will hang in the form of a perfect circular arc, provided only that it is less than a full semi-circle. That weighting will in turn indicate the ideal tapering of the arch for maximum structural stability. However, Libera's original specifications for his proposed arch give no indication that he was aware of the mathematics needed to determine the correct weighting or degree of tapering.[19]

Concluding remarks

The paintings and the book by Jasper Johns devoted to catenaries [2005] make it clear that the shape of a hanging chain evokes an esthetic response quite apart from any attempt to analyze that shape by mathematical means.

In the case of architecture, the need to combine esthetic values with mathematical and structural principles is one that dates back to antiquity, and is certainly apparent in the surviving Greek temples. In some cases, mathematics and esthetics were viewed as interchangeable. As is apparent from our discussion here, in the case of the Gateway Arch they are, if not equivalent, then at least inextricable. It is clear from his letters and his various explanations, that Saarinen was aware at a very early stage that the structural requirement for maximal stability translated into keeping the line of thrust aligned as nearly as possible with the slope of the Arch, which in turn determined the relevant shape. Having begun with that principle, and tried out the resultant shape, he turned it around, modified the pure catenary shape "according to the eye," and then left it to his (superb) engineering team to apply the mathematical principles that would produce the desired shape and satisfy the structural needs.

Viewed more broadly, the role of the architect is one that has provoked a surprisingly wide range of responses. At one extreme, there is the assessment reported in the book *Brunelleschi's Dome* by Ross King [2000: 157-158], that both ancient and medieval authors "assigned architecture a low place in human achievement, regarding it as an occupation unfit for an educated man." He cites both Cicero and Seneca in support of that view. At the other extreme, we have the view that "it is hardly surprising that, for the ancients, the image of the architect has demiurgic connotations" [Benvenuto 1991: Introduction, xix]. The author, Edoardo Benvenuto, goes on to say:

> In a famous dialogue of Paul Valéry[20] this "nearly divine" aspect is expressed in the following words of Phaedro to Socrates when speaking of his friend the architect Eupalinos:[21] "How marvelous, when he spoke to the workmen! There was no trace of his difficult nightly meditation. He just gave them orders and numbers.

(To which Socrates responds, "God does just that".)

A twentieth-century version of this story is reported by Bruce Detmers, whose duties as a young architect working with Saarinen on the Gateway Arch included long hours spent translating the engineers' formulas for Saarinen's Arch into long columns of numbers generated, in the absence of a modern computer, on an old Marchand calculator borrowed from the accounting department. These numbers were then passed on to those responsible for manufacturing the various components needed in the construction of the Arch.

Detmers recounts one incident that may resonate as a distant echo of Phaedro and Eupalinos. He was working away at two or three o'clock in the morning, when Saarinen wandered in, and asked what he was doing. When Detmers explained, Saarinen's laconic response was, "Keep going."

Additional remarks and open questions

We elaborate here on some of the points noted briefly in our earlier paper [2010], and conclude with some remaining questions.

1. It is often noted that viewers of the Gateway Arch tend very strongly to see it as being taller than wide, despite the fact that its height and width are both exactly equal at 630 feet. This effect is generally described as an optical illusion. But is it, in fact, an illusion?

While it is true that the *outside* curve, or extrados, is the same width as height, one's eye is as likely to fix on the inside curve, or intrados, or on something in between. In those cases, the curve one sees, or extrapolates, is in fact considerably taller than wide. The reason is simply that the arch is much thinner at the top than at the base, and that even further, the base width gets subtracted off twice between the outer and inner curves. In precise numerical terms, the inner curve is 615.3 feet high and 536.1 feet wide, so that the height is 15% greater than the width. It is likely the case that the impression of the Arch being taller than wide is partly due to this fact and partly a true optical illusion.[22]

2. It is clear that Saarinen was aware of Hooke's dictum relating the shape of a standing arch to that of a hanging chain, and that the goal was to build an arch where the direction of the line of thrust at each stage of the arch is as near as possible to the slope of the arch itself. He was further aware that the principle applied whether the chain was uniform or variable, forming a "weighted chain." In choosing the shape of a flattened catenary, rather than a pure catenary, he was able to satisfy two distinct desires simultaneously – first, to arrive at a shape more pleasing esthetically to him, since is was more rounded at the top, and second, to have the corresponding arch satisfy the structural needs of greater thickness and strength toward the bottom, while thinner and more slender near the top. If one compares both the pure and the flattened catenaries with a parabola, one sees that the parabola is even pointier at the top than the catenary (fig. 10).

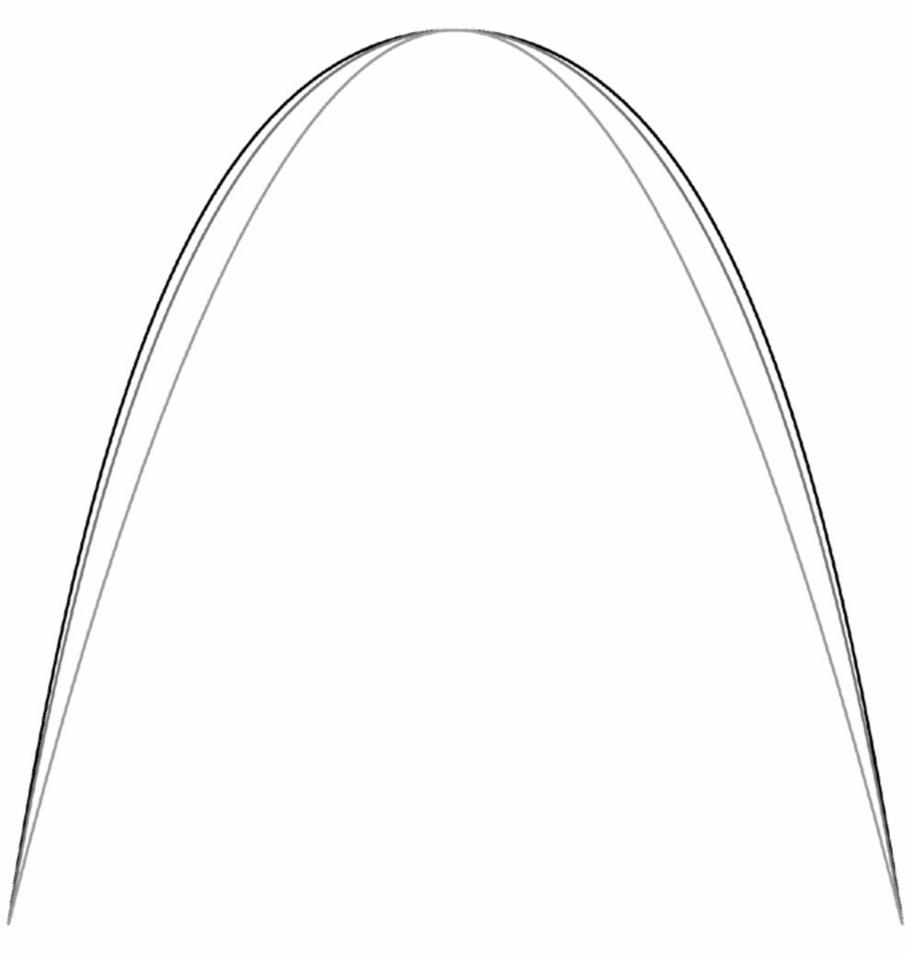

Fig. 10. Parabola, Catenary, and Flattened Catenary with Aspect Ratio 1:1

It is fortunate (or is it something deeper?) that Saarinen found the parabola less pleasing from a purely esthetic viewpoint, since following Hooke's dictum in that case would require greatest weight at the vertex, and therefore an arch that was thickest at the *top*, and slimming down toward the bottom—hardly what one would want from a structural point of view. In fact, it is clear that if one starts with a uniform chain that hangs in the shape of the catenary, then to make it pointier at the vertex one would want to increase the weight in the middle, while to make it rounder or flatter near the vertex, one would increase the weight near the ends.

Fig. 10 shows all three curves together, each with aspect ratio 1:1. The parabola is the innermost, pointiest curve, and the catenary the intermediate one, while the outer curve is a flattened catenary with the same degree D of flattening as that used in the Arch

3. There are at least two ways to arrive mathematically at the shape of a hanging chain. The first is the one we have been invoking, in which the object is to keep the thrust line always in the direction of the chain. The other is by using the method known as the calculus of variations to determine the shape of the chain that will give the lowest

center of gravity. Both methods yield the same result – that the chain should be in the shape of a (possibly weighted) catenary – and both methods make good sense physically. However, a strange consequence is that in the case of the arch, the shape of the catenary, while best possible from the point of view of the thrust line, is in a certain sense the *worst* possible structurally, since it will have the highest center of gravity, and therefore be the least stable. Partly to counter this paradoxical fact, and partly for other structural reasons, the engineers responsible for the Gateway Arch decided to fill in the space between the outer stainless steel surface and the inner carbon steel surface with concrete in the bottom 300 feet of the Arch. The weight of the concrete alone in the bottom 300 feet is more than double the total weight of all the steel in the Arch, while the underground foundation weighs more than double that of the above-ground concrete. The effect is to considerably lower the center of gravity, in addition to giving extra strength to the bottom half of the Arch, providing resistance to cantilever and torque effects from the wind, and limiting vibrations.

4. The fact that the method of construction changes abruptly roughly halfway up the Arch means that if one models the Arch as a weighted catenary, then the weighting will undergo a singularity at the point corresponding to the end of the concrete filling. It is as if the first 300 feet on both sides represent a pedestal, and the Arch proper sits on top of that. From the outside, of course, one sees a continuous curved surface, as well as a continuous variation in the size of the cross-sections. The principal question to which I have not found an answer is the reason for choosing the variation in the size of the cross-section in a way that makes the areas vary linearly with height. But the abrupt change in the weighting at the 300-foot level means that no single formula can be optimal for both of the parts – those lying above and below that level. Clearly the Arch is designed with a sizeable margin of safety beyond what the simple choice of a curve and weighting would provide. One would need such a margin for safety if for no other reason than the necessity for the two legs to stand on their own, like a pair of giant leaning towers, during the greater part of the construction. Only when they were over 500 feet high did they reach the point where they could be joined in mutual support.

5. The final questions to which I do not know the answer are: 1) Was the choice of 54 feet for the dimension of the triangular cross-section at the base of the monument an esthetic or a structural decision? (The dimension of 17 feet at the top, I have been told, was simply the smallest felt to be possible, allowing for the construction of an observation platform where visitors can stand up and admire the view from the top through the windows built into the Arch.); 2) What is the basis of the choice of the ratio Q_b / Q_v for the parameter R that enters into the basic equations (7) – (10)? I hope to return to these questions at a later occasion.

Acknowledgments

The most interesting and accurate mathematical discussions of the Gateway Arch that I am familiar with are those by William V. Thayer [1984], and a chapter (in Portuguese) of a volume of computational activities and projects designed to highlight applications of the calculus [Figueiredo et al. 2005]. I am particularly indebted to Bill Thayer for providing a wealth of documents as well as a number of significant leads in my investigation of this problem; also to Charles Redfield, one of the main engineers who worked on the Arch and who provided me with a number of relevant documents, Bruce Detmers, an architect who worked with Saarinen on the Arch, John Ochsendorf for enlightening discussions of his paper [Block et al. 2006] on arches and related matters, Jacques Heyman for prompt responses to a number of questions arising from his many books and articles about arches, Hélène Lipstadt and Jack Rees who offered a number of further valuable leads, and Robert Moore of the National Park Service, historian of the Gateway Arch. I also thank Jennifer Clark, archivist at the Old Court House in St. Louis, and Laura Tatum at the Saarinen Archive at Yale University

Notes

1. One can also treat a cord from which are hung a series of literal weights, in which case the cord will take a kind of polygonal shape. However, we shall not consider that case here, since it is not too relevant to the curve of the Gateway Arch.
2. This and all further references to statements by Bruce Detmers come from conversations with him in March 2008.
3. Robert Hooke had the great misfortune to be a contemporary of Isaac Newton, and on two counts. First of all, Newton's brilliance and monumental achievements simply overshadowed any and all potential rivals among scientists. But perhaps even more so, because Newton developed one of his notorious lifetime grudges against Hooke, and did everything possible – which was quite a bit – to denigrate and belittle Hooke and his achievements.
4. See also [Huerta 2006]. The issue here is that although Galileo is precisely correct in his great insight that one cannot scale up any structure an arbitrarily large amount, since the weight goes up as the cube of the scaling factor and the cross-sectional area of the supports – say columns for a building, legs of an animal, or trunk of a tree – increases by only the square, nevertheless he is "wrong" not to note that on the scale of normal buildings, strength is not the key issue, but stability, as in Hooke's dictum, and in that case what counts is the geometrical shape, invariant under scaling.
5. Strictly speaking, the term used was the Latin word, *catenaria*, and the English word "catenary" did not occur until somewhat later; this will be discussed shortly. An excellent and detailed history of the "catenary problem" is given by Clifford Truesdell in The Rational Mechanics of Flexible or Elastic Bodies, 1638-1788; Introduction to *Leonhardi Euleri Opera Omnia*, Vol. X et XI Seriei Secundae, Lausanne, Orell Füssli Turici 1960; see also further references in footnote 7 below. (Surprisingly, Truesdell also fails to cite Galileo's correct description of the catenary as an approximation to a parabola.)
6. *"Tout homme est pierre; tout caillou est homme; donc tout caillou est pierre.*" From a letter of August 11, 1697; Jacob Bernoulli, *Opera*, pp. 829-839; reprinted in *Die Streitschriften von Jacob und Johann Bernoulli.* Birkhäuser Verlag 1991, pp. 356-364 (see p. 361). See also pp. 1-114 of this volume for an overview in English of the contents, and pp. 117-122 for a list of "The Polemic Writings of Jacob and Johann Bernoulli on the Calculus of Variations" that are reproduced in the book.
7. Other names that should be mentioned in this brief history are Philippe de la Hire and David Gregory. De la Hire's *Traité de méchanique* from 1695 addresses the question of how to distribute weights along a chain to attain "a figure curved the way you wish it to be" (proposition 123), and he makes explicit the connection between arches and variably weighted chains. David Gregory discovered independently the relation between weighted chains and arches and described them in a letter later published in the *Philosophical Transactions* in August 1697. For more on both of these major contributions, see [Benvenuto 1991: 321-329]. Another good historical reference is [Bukowski 2008] (the usual misleading statements about Galileo are repeated here, but it is instructive to see how early they originated and were propagated by successive generations).
8. At least that is the case for the dozens of articles, as well as congratulatory telegrams that are on file in the Saarinen Archives at YaleUniversity.
9. Copies of Grant's letter and Saarinen's response are in the Saarinen archive. It is the only place I have found where Saarinen describes his winning design with as much specificity.
10. Quoted in [Crosbie 1983].
11. Whether this formula (or, for that matter, any other formula) is displayed inside the Arch as claimed, is something I have not been able to determine. It is not impossible, since the precise

shape of the Arch was modified so many times during the planning process, but it is unlikely, because the decision not to use an exact catenary was made very early – long before construction began.

12. The precise conditions for the graph of a curve to be representable as a weighted catenary are that it be convex, and that it makes a non-zero angle with the vertical at its endpoints. See [Osserman 2010].
13. Johann Bernoulli was already aware of the connection between arches and a chain of variable weight in 1698; see, for example, [Benvenuto 1991: Pt. II, 327].
14. There are 71 such sections on each leg of the arch, according to the specifications.
15. This is further complicated by a note on the blueprints that for the top 41 sections, the "exterior skin to be curved into a continuous smooth surface." This corresponds to roughly the upper half of the Arch, where the curvature is the greatest. It is also above the 300-foot level where there is another transition in the mode of construction, as we discuss more fully later on; below that level the stainless steel outer skin is backed by a layer of concrete.
16. Both of these examples are described in fascinating detail in the book *The Tower and the Bridge* [Billington 1985].
17. George B. Hartzog, Jr. was Director of the National Park Service at the time. See the chapter on the Gateway Arch in [Hartzog 1988], especially pp. 52-56.
18. See, for example, [Coir 2006]: "Milles strongly advised Saarinen to drop his initial plans to fashion the arch with a quadrilateral section in favor of a more sculpturally pleasing triangular section. Eero's failure publicly to recognize Milles's contribution to the monument angered the Swedish sculptor, who thereafter severed contact with Saarinen" [2006: p. 41 and footnote 40, p. 43].
19. On the other hand, some of Libera's sketches for the arch show a radical thickening near ground level, which would seem to indicate at least an intuitive understanding of the need for a suitable weighting to support the shape of a circular arc. From an esthetic point of view that thickening may be seen as producing a more dramatic effect for Libera's arch than for Saarinen's, or else as far less graceful. (Perhaps both.)
20. Benvenuto is referring to Valéry's "Eupalinos ou l'architecte" [1960: 83].
21. Eupalinos was the engineer to whom is ascribed (by Herodotus) the construction of the famous 3400 foot long tunnel on Samos, excavated from both ends, meeting in the middle.
22. A number of experiments seem to confirm that we tend to consistently overestimate lengths in the vertical direction over equal ones that are horizontal. See for example [Wolfe et al. 2005: 967-979] and further references given there.

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