Green's Conjecture on Free Resolutions and Canonical Curves

by David Eisenbud

David Hilbert, in his work on Invariants, established a fundamental link between the algebra of polynomial rings and the geometry of complex projective space $\mathbf{P}^n$. His Nullstellensatz shows that the correspondence taking an algebraic variety $X \subset \mathbf{P}^n$ to the ideal I_X of polynomials vanishing on X in $S := \mathbf{C}[x_0, \ldots, x_n]$ is one-to-one and onto the set of prime ideals.

Hilbert took another big step in describing a way of getting geometric invariants from the algebra. Let $H_X(d) = \dim_{\mathbf{C}}(S/I_X)_d$ is the dimension of the vector space of homogeneous polynomials of degree d modulo those vanishing on X. Hilbert proved that the function $H_X(d)$ is equal, for large d, to a polynomial $P_X(d)$ in d. The coefficients of $P(d)$ are geometrically significant numbers. For example, if X is a compact Riemann surface then the constant term $P(0)$ is equal to $1 - g$, where g is the genus (number of holes) of the Riemann surface.

In showing that H_X is eventually equal to a polynomial, Hilbert defined a much finer invariant, the free resolution of $S_X := S/I_X$. The idea is to take a generator (1) for S_X as a module over S; then the relations it satisfies (homogeneous generators $f_1, \ldots, f_s$ for I_X); then the module of relations that these satisfy (called syzygies of I_X: these are vectors $g_1, \ldots, g_s$ such that $\sum g_i f_i = 0$); then the module of syzygies of the syzygies of I (called second syzygies of I); and so on.

If I_X happens to be generated by 1 polynomial f_1 then I_X has only trivial syzygies. This would always be the case if $n = 1$. Hilbert generalized this remark to arbitrary n with his "Syzygy Theorem", which I still find astonishing. It (or rather a special case) says that for any X the d-th syzygy module of I_X has only trivial syzygies for $d \geq n - 1$. Equivalently, there is a *finite* exact sequence of graded modules

$$\mathbf{F}: \qquad 0 \to F_m \xrightarrow{d_m} \cdots \longrightarrow F_1 \xrightarrow{d_1} S \longrightarrow S_X \longrightarrow 0,$$

where each F_i is a free module over S, called a free resolution of S_X.

To prove from this that the Hilbert function H_X is eventually a polynomial is easy: H_X is the alternating sum of the Hilbert functions $H_{F_i}(d) = \dim_{\mathbf{C}}(F_i)_d$. The module F_i itself, being free, is a direct sum of copies of S with generators in various degrees. If we write $S(-a)$ for the free module of rank 1 with generator in degree a, then we have $\dim_{\mathbf{C}}(S(-a))_d = \binom{n-a+d}{n}$. This binomial coefficient is equal to a polynomial in d for all $d \geq a - n$, proving that the Hilbert function is eventually polynomial (and making it interesting to compute a bound on the degrees a that occur; but this belongs to another story.)

For a very simple example, consider a linear subspace $X \subset \mathbf{P}^n$ of codimension 3, defined by the vanishing of $I_X = (x_0, x_1, x_2)$. A free resolution of $S_X = S/(x_0, x_1, x_2)$ has the form

$$0 \longrightarrow S(-3) \xrightarrow{\begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}} S(-2)^3 \xrightarrow{\begin{pmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{pmatrix}} S(-1)^3 \xrightarrow{(x_0 \;\; x_1 \;\; x_2)} S \longrightarrow S_X \longrightarrow 0.$$

(This example appears in Hilbert's 1890 paper, and is a special case of what is now called the Koszul complex.) Thus

$$H_X(d) = \binom{n+d}{n} - 3\binom{n-1+d}{n} + 3\binom{n-2+d}{n} - \binom{n-3+d}{n}.$$

If, in computing a resolution, we always choose a *minimal* number of generators at each step, then we get a *minimal free resolution,* of S_X, and it is not hard to show that this is unique up to isomorphism. In particular, the degrees of the generators of the free modules are determined by $X \subset \mathbf{P}^n$, and this collection of numbers gives a finer invariant than the Hilbert function or polynomial. What geometric significance does this invariant have?

The most interesting cases to study are those in which the embedding of X in $\mathbf{P}^n$ depends only on the intrinsic geometry of X, so that the invariants we get will be invariants of that geometry. For example we can use a basis of the holomorphic sections of the cotangent bundle of a Riemann surface X of genus $g \geq 3$ to define a canonical map from X to $\mathbf{P}^{g-1}$. This map will be an embedding except in the "degenerate" hyperelliptic case, when X is a double cover of $\mathbf{P}^1$. The degrees that appear in the minimal free resolution of S_X, when X is canonically embedded in this way, are thus invariants of the geometry of X.

Perhaps the most important invariant of a Riemann surface after its genus is its *Clifford index*, a number that measures how special the surface is from the point of view of having low degree mappings to small projective spaces. For example a Riemann surface has Clifford index 0 if it admits a two-to-one mapping to $\mathbf{P}^1$; it has Clifford index ≤ 1 if it admits a three-to-one mapping to $\mathbf{P}^1$ or an embedding in $\mathbf{P}^2$ as a curve of degree 5. In general, it is a good approximation to the truth to think that a Riemann surface has Clifford index $\leq c$ if it admits a $c+2$-to-one mapping to $\mathbf{P}^1$.

Mark Green [1] conjectured that one could read the Clifford index of a Riemann surface X from the minimal resolution of S_X when $X \subset \mathbf{P}^{g-1}$ is canonically embedded. More precisely, if the differentials $d_2, \ldots d_{t-1}$ are represented by matrices of linear forms but d_t is not, then the Clifford index $c(X)$ should be precisely t. There is an easy geometric reason why $t \leq c(X)$, and Schreyer, Voisin, and others proved special cases including all cases for $g \leq 8$, but the inequality $t \geq c(X)$ has remained obscure.

However, there have been two recent breakthroughs in this subject, one by Montserrat Teixidor I Bigas [2], and one by Claire Voisin [3]. Together they show the conjecture is right at least "most of" the time:

Theorem. *Except for the case when g is odd and $c = (g+3)/2$ the set of Riemann surfaces of genus $g \geq 3$ and Clifford index c that satisfy Green's conjecture contains an open set (in the moduli space of such Riemann surfaces.)*

Much more is known about this conjecture than I have been able to indicate here. The introductions to the papers listed above will give a start on the literature. My manuscript-in-progress [0], which will probably appear in the Springer Graduate Texts in Math series in 2003, gives a more extended account of how geometry and syzygies interact.

References

[0] David Eisenbud: The Geometry of Syzygies. For an almost-final version of this manuscript, frequently updated, see `http://www.msri.org/people/staff/de/ready.pdf.` I'd be very interested in getting feedback on it before December 2002!

[1] Mark Green: Koszul cohomology and the geometry of projective varieties. J. Differential Geom. 19 (1984), no. 1, 125–171.

[2] Montserrat Teixidor I Bigas: Green's conjecture for the generic r-gonal curve of genus $g \geq 3r - 7$. Duke Math. J. 111 (2002), no. 2, 195–222.

[3] Claire Voisin: Green's generic syzygy conjecture for curves of even genus lying on a K3 surface. Available at `http://arxiv.org/math.RA/0205330`