Representations of Motion Groups

Joint with:

- Gustafson, Kimball, Zhang
- Bullivant, Kimball, Martin
- Damiani, Martin
- Kadar, Martin, Wang

NSF, MSRI, Simons Foundation: Thanks!
Motivation:

- 2D picture: $B_n$ reps are key!
  - Anyon statistics in 2D Topol. phases.
  - Density of unitary reps $\leftrightarrow$ Universal Top. Quantum Computation.
  - Link/3-manifold invariants via Trace.
  - $(2+1)$TQFTs: genus 0 part.

- 3D picture: Motion Groups.
  - Loop excitations in 3D TPMs?
  - Any useful Q.C. Models?
  - Interesting invariants?
  - $(3+1)$TQFT “Canary in the Coal Mine”
Motion Groups.

M oriented

N oriented, cpt

M oriented

A motion of \( N \) in \( M \) is an ambient isotopy \( f_t(x) \) of \( N \) in \( M \)

\( f_0(x) = \text{id}_M \) & \( f(\mathbb{N}) = \mathbb{N} \)

as an oriented submanifold.

(\text{N.b.} \ f_t|_\mathbb{N} \neq \text{id}_\mathbb{N} \text{ in gen.})

If \( f_t(\mathbb{N}) = \mathbb{N} \ \forall t \), \( f \) is stationary.

\( f \equiv f \)

if \( f \circ f \sim f \) a stationary motion.

\( \mathcal{M}(M,N) \): motions/\sim
More details

0 $H^c_c(M,N)$ homeos (or diff'ns) of $M$ w/ 
- $M$ fixed ptwise
- $N$ fixed setwise
- orientation on $M$ & $N$ preserved.

0 Motions are paths $f$ in $H^c_c(M)$ st 
- $f_0 = \text{id}_M$
- $f_1 \in H^c_c(M,N)$

\[ \mathcal{U}(M,N) = \pi_1(H^c_c(M), H^c_c(M,N); \text{id}_M) \]

0 $H^+(M,N) : = \pi_0(H^c_c(M,N), \text{id}_M)$ orientation preserving

0 $\mathcal{U}(M,N) \xrightarrow{d} H^+(M,N)$ $d([f]) = [f_1]$. 

- $d$ is iso if $M = \mathbb{R}^3$
- $d$ surj. if $M = S^3$, $\ker(d) \cong \mathbb{Z}_2$. 
Remarks & Subtleties

- $M \text{ cpt? } \partial M \neq \emptyset$ orientation, diff? 

- Focus on $M^2, M^3$: \boxed{$S^3, D^3, R^3$ or $S^2, D^2, R^2$}

- $M(M, N) \rightarrow \text{Aut}(\pi_1(M-N))$ or $\text{Out}(\pi_1(M-N))$
  can be useful for getting a presentation.

- Few presentations available.
  - $\sqcup Q, (\sqcup \odot) \sqcup (\sqcup \odot)$ [Damiani-Kamada]
  - $\sqcup$ [Belliingeri-Bodin]
  - Tors Links [Goldsmith, Qin-Wang]
Example I. The Braid Group

\[ B_n := \mathcal{M}(D^2, p) \]
\[ |P| = n. \]

\[ \sigma_i \rightarrow \begin{vmatrix} 1 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \]

Motions of points in a disk.

- Close to Hurwitz' 1891 formulation...
- Artin's presentation:
  
  \[ (B1) \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \]
  \[ (B2) \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for } |i - j| > 1, \]
Example II (McCool, Finn-Rourke-Rimanyi, ...)

$LB_n = M(B^3, S^1 \ast S^1 \ast \ldots \ast S^1) \cong B_n \ast S_n / \langle L_0, L_1, L_2 \rangle$

The braid relations:
(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| > 1$,

the symmetric group relations:
(S1) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$
(S2) $s_i s_j = s_j s_i$ for $|i - j| > 1$,
(S3) $s_i^2 = 1$

and the mixed relations:
(L0) $\sigma_i s_j = s_j \sigma_i$ for $|i - j| > 1$
(L1) $s_i s_{i+1} \sigma_i = \sigma_{i+1} s_i s_{i+1}$
(L2) $\sigma_i \sigma_{i+1} s_i = s_{i+1} \sigma_i \sigma_{i+1}$

or $\sigma_i \sigma_{i+1} s_i = s_i \sigma_i \sigma_{i+1}$

but not both!
Example III (Bellingeri-Bodin)

$NB_n : \mathcal{M}(S^3, \mathcal{N})$

(B1) $\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$
(B2) $\sigma_i \sigma_j = \sigma_j \sigma_i$ for $|i - j| \neq 1 \pmod{n}$,
(N1) $\tau \sigma_i \tau^{-1} = \sigma_{i+1}$ for $1 \leq i \leq n$
(N2) $\tau^{2n} = 1$

Here indices are taken modulo $n$, with $\sigma_{n+1} := \sigma_1$ and $\sigma_0 := \sigma_n$. 
Example IV (Goldsmith, Qiu-Wang)

Torus Links $TL(n_p, n_q)$

- $\sigma_i$ interchange $i$th component & $(i+1)$st.
- $r_i$ rotate $i$th component by $\frac{2\pi}{p}$.
- $\sigma_i$ satisfy Braid rels.
Remarks on Reps. of Motion Gps

- Explicit reps from (3+1) TQFT are not easy.

- In many cases, $B_n \subseteq \mathcal{M}(H^3, L)$

- Might hope for invariants of surfaces in $R^4$ via Markov traces, generalized.

- Physics applications! Levin-Wang

PRL 2014
Representations of $B_n$:

- **1936**: Burau

\[
\begin{bmatrix}
1 & 0 \\
0 & 1 \\
\end{bmatrix}
\]

- **1980s**: Yang-Baxter equation

  \[ R_1 = R_2 = I \otimes I, \quad R_1 R_2 R_1 = R_2 R_1 R_2 \]

- **1980s**: Jones, Birman-Wenzl, others...

  Towers of f.d. quotients of $H[B_n]$:

  \[ T_{L_n}(t), \quad H_n = C(q) \left[ B_n \right]/\langle (\sigma_i t + 1)(\sigma_i - q) \rangle \]

  BMW$_n(r, q)$
Representations of $B_n$ (Cont)


- 1990s Braided fusion category

  $\Psi: C[B_n] \rightarrow \text{End}(X^{\otimes n})$, $\sigma_i \mapsto \frac{1}{n} \otimes \mathbb{C}_{x,x} \otimes I_x$

  $B_n$ acts on $\bigoplus \text{Hom}(Y, X^{\otimes n})$, $Y \in \text{Im}(\lambda)$

- 1989 Jones-Goldschmidt:

  Metaplectic reps $A_n = \mathbb{C}\langle u_i : u_i u_{i+1} u_i = q^2 u_{i+1} u_i u_i = q = 1, [u_i, u_j] = 1 \ (i-j)/2 \rangle$

  $\sigma_i \mapsto \frac{1}{\sqrt{k}} \sum_{j=1}^{k^2} q^{ij} u_j^i$ defines a rep into $A_n$. 
Representations of $B_n$ (even more!)

Inspired by metaplectic reps.

Fix finite gp $G$ and a bihom. $\alpha: G \times G \to \mathbb{Z}_m$

be $q^n = 1$. Iterated twisted tensor power

$$A_n(G, \alpha) = \mathbb{C}[G] \otimes_{\alpha} \mathbb{C}[G] \otimes_{\alpha} \cdots \otimes_{\alpha} \mathbb{C}[G]$$

Look for

$$r_i = \sum_{g \in G} f(g) g_i$$

satisfying

$$B_n \text{ reps, possibly in } A_n(G, \alpha) / \mathbb{C}$$

- Metaplectic: $G = \mathbb{Z}_2$
- $G = Q_8 \to \text{Quaternionic } B_n \text{ reps.}$
- $G = \mathbb{Z}_p \times \mathbb{Z}_p \text{ factors...}$
Categorical connections

0 Rieutcurr: S. S. (3+1) TQFT $Z$, $M^4$ spinless
then $Z(M^4)$ depends on classical stuff
($\pi_1, X, \ldots$)

0 Liange Cheng: $\mathbb{Z}$ reps from $X, Y \in \mathcal{BFC}$ if

$x \otimes y = \bigoplus Z_i$, $Z_i$ bosons/fermions. In fact,
non 2-dim. braiding on $(x \otimes y)^\otimes$

$\Rightarrow \mathcal{E}[X, Y]$ is weakly integral, so
finite images, probably
Categorical construction: Dijkgraaf-Witten Theory

Qiu-Wang provide evidence that (Conj) reps of motion gps from $DW_6^{3+1}$ are determined by those from $DW_6^{2+1}$.

E.g. Torus Links with labels pure fluxes, (& mapping class gps of closed mfd's).

0 How general is this?
Loop BFCs?

Very Naive guess: \( X \in \mathcal{V} \text{ BFC}, \quad \sigma_{X,X} \in \text{End}(X \otimes 2) \).

A symmetric braiding \( S_{X,X} \in \text{End}(X \otimes 2) \) also?

Unusual by results of Nikshych. X

Loop Braided Vector spaces? \([K \text{id}ar, \text{Martin}, R, Wang]\)

\((R, S, V) R, S \in \text{End}(V \otimes 2) \) solns to YBE

\[ \text{st.} \quad I_{Bn} \rightarrow \text{Aut}(V \otimes n) \quad \sigma_i \rightarrow R_i, \quad S_i \rightarrow S_i. \]

Thm: If \( R \) is of group type (i.e., YD-module)

\[ S(v \otimes w) = w \otimes v \text{ works! (Essentially DW).} \]

Otherwise, may be not.
<table>
<thead>
<tr>
<th>Extensions?</th>
<th>$L_{Bn}$</th>
<th>$N_{Bn}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$B_n$ rep</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$B_n$ rep</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Lawrence-Kramer</td>
<td></td>
<td>✓</td>
</tr>
<tr>
<td>Lash</td>
<td></td>
<td>✓</td>
</tr>
</tbody>
</table>
| Arb. completely reducible |          | + many counterexs. | central with |}
| $B_3$ irrep. dim $< 5$     |          | ✓        |
| Gaussian, Quaternionic    |          | ✓        |
| Categorical constructions  |          | ✓        | YοX on if $C_{X,Y}^\circ C_{Y,X} = id.$
Finite dim' quotients? [Damiani - Martin - R]

\[ C[IB_n] / \langle (\sigma_i - 1)(\sigma_i + t) \rangle \]

However, extended Burau is a rep.

\[ \sigma_i \mapsto M_i = \begin{bmatrix} 1 & t & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad s_i \mapsto \sigma_i \bigg|_{t=1} = P_i \]

These satisfy:

\[ (M_i - I)(P_i - 1) = (P_i - 1)(M_i + t) = 0. \]

Set \( LH_n := \mathcal{I}B_n / \langle (\sigma_i - 1)(\sigma_i + 1), (\sigma_i - 1)(s_i + 1), (s_i - 1)(\sigma_i + t) \rangle \)

Finite dim'?
Properties of $\text{LH}_n$ (Loop Hecke algebra)

- Not semisimple!
- Admits a local rep: $R = \begin{bmatrix} 1 & 1-t & t & 0 \\ 1 & 0 & 0 & t \\ 0 & 1 & -t & 0 \\ 0 & 0 & 1 & -t \end{bmatrix}$

$S = R|_{t=1}$.

$\sigma_i \rightarrow I_2 \otimes R \otimes I_2^{n-1-i}$

$S_i \rightarrow I_2^{i-1} \otimes S \otimes I_2^{n-i-1}$

Loop Burau-Rittenberg rep.
Structure of $LBR_n$ algebra: $\langle R_i, S_i \rangle$.

- $\dim(V_i) = (\frac{n}{i})^2$ (simple)
- $\dim(W_i) = \binom{n}{i} \binom{n}{i+1}$
- Jacobson Radical $\bigoplus W_{i,i+1}$
- $LBR_n/J(LBR_n) \cong \bigoplus A_i$
- Bratteli diagram: Pascal's Triangle
Outlook / Future directions

0 Non-s.s. is a feature, not a failing?
0 A more robust categorical approach is needed.
0 Other f.d. quotients of Motion Group algebras?
0 Topological invariants (of surfaces in $\mathbb{M}^4$)?
Markov Trace on $LH_n$?

0 $\text{Tr}(ab) = \text{Tr}(ba)$  $a, b \in LH_n$

0 $\text{Tr}(a \sigma_n) = \text{Tr}(a) \alpha$  Then adjust by

0 $\text{Tr}(a s_n) = \text{Tr}(a) \beta$

0 $\text{Tr}(1) = 1$

Relations $\Rightarrow (\alpha - t\beta)(\alpha - \beta) = 0$

\[ (-t, -1) \quad (1, 1) \]

Leads to trivial invariant...