# Effective Hausdorff Dimension and Applications

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# Abstract

The Hausdorff Dimension of a set of real numbers A is a numerical indication of the geometric fullness of A. Sets of positive measure have dimension 1, but there are null sets of every possible dimension between 0 and 1.

Effective Hausdorff Dimension is a variant which incorporates computability-theoretic considerations. By work of Jack and Neil Lutz, Elvira Mayordomo, and others, there is a direct connection between the Hausdorff dimension of A and the effective Hausdorff dimensions of its elements. We will describe how this point-to-set principle works and how it allows for novel approaches to classical problems in Geometric Measure Theory.

# Lebesgue Measure

For convenience we will work in Cantor space $\mathcal{C}$, wherein the points are infinite binary sequences $x \in 2^\omega$ and a basic open set $B(\sigma)$ consists of all extensions of a particular finite binary sequence $\sigma \in 2^{<\omega}$.

We obtain Lebesgue measure $\lambda$ on $\mathcal{C}$ by setting $\lambda(B(\sigma)) = 1/2^{|\sigma|}$, where $|\sigma|$ denotes the length of $\sigma$, and applying Lebesgue's method of extension. Then, when $A$ is measurable,

$$\lambda(A) = \inf \left\{ \sum_{k=1}^{\infty} \lambda(B(\sigma_k)) : \begin{array}{l} (\sigma_k)_{k\in\mathbb{N}} \text{ is a sequence from } 2^{<\omega} \\ \text{with } A \subseteq \bigcup_{k=1}^{\infty} B(\sigma_k) \end{array} \right\}.$$

# Regularity of Lebesgue Measure

**Remark**

*If $A$ is measurable, then*

$$\lambda(A) = \inf\{\lambda(O) : O \textit{ is open and } A \subseteq O\}$$
$$= \sup\{\lambda(C) : C \textit{ is closed and } C \subseteq A\}$$

In other words, the measure of $A$ is carried by the measures of its closed subsets.

# Randomness

## formulated by measure

**Definition**

A sequence $x$ is *Martin-Löf random* iff it does not belong to any effectively-null $G_\delta$ set. Precisely, if $(O_k : k \in \mathbb{N})$ is a uniformly computably enumerable sequence of open sets such that for all $k$, $O_k$ has measure less than $1/2^k$, then $x \notin \cap_{k\in\mathbb{N}} O_k$.

This is not mysterious: Identify a family of sets of measure 0, and say that $x$ is random if it does not belong to any set in the family.

# Randomness

formulated by compressibility

**Definition**

- For $\sigma \in 2^{<\omega}$, $K(\sigma)$ is the length of the shortest program which outputs $\sigma$ and then halts, in a universal prefix-free listing of programs.
- A sequence $x \in 2^\omega$ is *algorithmically incompressible* iff there is a $C$ such that for all $\ell$, $K(x \upharpoonright \ell) > \ell - C$, where $K$ denotes prefix-free Kolmogorov complexity.

This is also not mysterious: Say that $x$ is incompressible when for all $\ell$, it takes $\ell$ plus a constant number of bits of information to describe $x \upharpoonright \ell$.

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### Theorem (Schnorr 1973)

*$x$ is Martin-Löf random iff it is algorithmically incompressible.*

# Random Sequences and Closed Sets

A closed set $C$ in $2^\omega$ can be represented as the set of infinite paths in a subtree $T$ of $2^{<\omega}$. (The terminal nodes of $T$ index the basic open sets that constitute the complement of $C$.)

When $T$ is computable, then $C$ is a $\Pi^0_1$ class. Otherwise, $C$ is $\Pi^0_1$ relative to $T$.

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When $T$ is computable, then $C$ is a $\Pi^0_1$ class. Otherwise, $C$ is $\Pi^0_1$ relative to $T$.

**Theorem (Folklore)**

- *If $C$ is $\Pi^0_1$ relative to $T$, then $C$ has positive measure iff $C$ has an element which is Martin-Löf random relative to $T$.*
- *An arbitrary set $A$ has positive measure iff for all $T$ there is an element of $A$ which is Martin-Löf random relative to $T$.*

We have a *point-to-set principle* for measure: prove that $A$ has arbitrarily random elements and conclude that $A$ has positive measure.

# Hausdorff Dimension

Define a family of outer measures, parameterized by $d \in [0,1]$. For $A \subseteq 2^\omega$,

$$\mathcal{H}^d(A) = \lim_{r\to 0} \inf \left\{ \sum_i \frac{1}{2^{|\sigma_i|\, d}} : \begin{array}{l}\text{there is a cover of } A \text{ by balls} \\ B(\sigma_i) \text{ with } 1/2^{|\sigma_i|} < r\end{array} \right\}.$$

**Definition**

The *Hausdorff dimension* of $A$ is as follows.

$$\begin{aligned}\dim_{\mathrm{H}}(A) &= \inf\{d \geq 0 : \mathcal{H}^d(A) = 0\} \\ &= \sup\left(\{d \geq 0 : \mathcal{H}^d(A) = \infty\} \cup \{0\}\right)\end{aligned}$$

# Frostman's Lemma

**Theorem (Frostman 1935, Carleson 1967)**

*For $A$ an analytic subset of $2^\omega$,*

$$\dim_H(A) = \sup\left\{ s : \begin{array}{l} \text{there is a Borel measure } \mu \text{ such that } \mu(A) > 0 \\ \text{and for all } \sigma \in 2^{<\omega},\ \mu(B(\sigma)) \leq \left(\frac{1}{2^{|\sigma|}}\right)^s \end{array} \right\}$$

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**Corollary**

*If $A$ is an analytic subset of $2^\omega$ and $\dim_{\mathrm{H}}(A) = d$, then for every $s < d$ there is a closed set $C_s \subseteq A$ such that $s \leq \dim_H(C) \leq d$.*

In other words, the Hausdorff dimension of analytic $A$ is carried by the dimensions of its closed subsets.

# Effective Hausdorff Dimension

## formulated by measure

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**Definition**

- For $A \subseteq 2^\omega$, define $A$ has *effective s-dimension Hausdorff measure 0* iff there is a uniformly computably enumerable sequence of open sets $O_i = \bigcup_j B(\sigma_{i,j})$ such that for each $i$, $A \subseteq O_i$ and $\sum_j (1/2^{|\sigma_{i,j}|})^s < 1/2^i$.
- The *effective Hausdorff dimension* $\dim_H^{eff}(A)$ of $A$ is the infimum of those $s$ such that $A$ has effective $s$-dimension Hausdorff measure 0.

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- The *effective Hausdorff dimension* $\dim_{\mathrm{H}}^{\mathit{eff}}(A)$ of $A$ is the infimum of those $s$ such that $A$ has effective $s$-dimension Hausdorff measure 0.

### Remark

- *For all A*, $\dim_{\mathrm{H}}(A) \leq \dim_{\mathrm{H}}^{\mathit{eff}}(A)$
- *If x is Martin-Löf random then* $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\}) = 1$.

# Effective Hausdorff Dimension

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**Definition**

A sequence $x \in 2^\omega$ is *algorithmically compressible by a factor of s* iff there is a $C$ such that there are infinitely many $\ell$ such that $K(x \upharpoonright \ell) \leq s\,\ell - C$, where $K$ denotes prefix-free Kolmogorov complexity.

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**Theorem (Mayordomo 2002)**

*For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the $s$ such that $x$ is algorithmically compressible by a factor of $s$.*

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**Theorem (Mayordomo 2002)**

*For any $x \in 2^\omega$, $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$ is the infimum of the $s$ such that $x$ is algorithmically compressible by a factor of $s$.*

- We will abbreviate and write $\dim_{\mathrm{H}}^{\mathit{eff}}(x)$ for $\dim_{\mathrm{H}}^{\mathit{eff}}(\{x\})$.
- We can relativize to a real $z$ and write $\dim_{\mathrm{H}}^{\mathit{eff}(z)}(x)$.

# Frostman's Lemma Revisited

**Theorem (Reimann 2008)**

*Suppose that* $\dim_{\mathrm{H}}^{\mathit{eff}}(x) = d$. *For all* $s < d$, *there is an* $s$*-regular Borel measure* $\mu$ *such that* $x$ *is Martin-Löf random for the measure* $\mu$.

# Point-to-Set for Hausdorff Dimension

**Theorem (J. Lutz and N. Lutz 2017)**

*For $A \subseteq 2^\omega$, the Hausdorff dimension of $A$ is equal to*
*the infimum over all $B \subseteq \mathbb{N}$*
*of the supremum over all $x \in A$*
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*of the effective-relative-to-$B$ Hausdorff dimension of $x$.*

Notice that there is no restriction on $A$ in the above theorem.

# Co-analytic Sets

## Consistency results

We will look at these phenomena in Gödel's universe of constructible sets $L$, consisting of those sets obtained from the empty set and transfinitely iterating first order definability.

In what follows, assume that every set is constructible, i.e. $V = L$.

# Co-analytic Sets

## Working in $V = L$

**Definition**

Define $P$ by

$$P = \left\{ x : \begin{array}{l} x \text{ can compute a representation of the ordinal at} \\ \text{which } x \text{ is constructed} \end{array} \right\}$$

**Theorem (Original reference unknown to me)**

- *$P$ is co-analytic.*
- *$P$ is not countable.*
- *$P$ has no perfect subset.*

# Co-analytic Sets

Working in $V = L$

**Theorem**

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Consequently, the following are consistent with *ZFC*

- The Hausdorff dimension of co-analytic sets is not carried by their closed subsets.
- The Frostman/Carleson Theorem does not extend further to co-analytic sets.

# Applying Point-to-Set Reasoning

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**Step 1.** There is an infinite computable set $S \subseteq \mathbb{N}$ such that for all $z$ and for all $x$, if $x$ is Martin-Löf random relative to $z$ and $y$ is equal to $x$ at all places not in $S$ then $\dim_{\mathrm{H}}^{eff(z)}(y) = 1$.

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In fact, $S$ could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of $y$ relative to $z$.

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In fact, $S$ could be the iterated powers of 2. To verify the claim, use Mayordomo's theorem and estimate the compressibility of $y$ relative to $z$.

**Step 2.** By the Lutz and Lutz theorem, it is sufficient to show that for every $z$ there is a $y$ in $P$ such that $\dim_{\mathrm{H}}^{eff(z)}(y) = 1$.

# Applying Point-to-Set Reasoning

## Working in $V = L$

**Step 3.** Suppose that $z \in 2^\omega$ is given.

- Let $x$ be Martin-Löf random relative to $z$.
- Let $m \in P$ be such that $m$ can compute $x$ and $z$.
- Let $y$ be the result of replacing the bit values of $x$ on the elements of $S$ by the bit values of $m$.

Then, $m$ can compute the ordinal at which $y$ is constructed and $y$ can compute $m$. Thus, $y \in P$.

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Then, $m$ can compute the ordinal at which $y$ is constructed and $y$ can compute $m$. Thus, $y \in P$.

**Step 4.** Conclude, $\dim_H(P) = 1$, as required.

# Comments

- J. Lutz, N. Lutz and Don Stull have other applications of effective Hausdorff dimension within Geometric Measure Theory.
- This mode of argument is in an early phase. It would be interesting to see whether/how it develops.

The End