

q -Orthogonal Polynomials and Their Connection to RHPs

Tomas Lasic Latimer (Program Associate)

University of Sydney

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q -Orthogonal Polynomials $\sum_{k=0}^{\infty} P_n(q^k)P_m(q^k)w(q^k)q^k = \gamma_n\delta_{n,m}$

Moving from classical (Hermite, Jacobi, Laguerre) to q discrete:

$$\frac{d}{dx} \rightarrow D_q, \text{ where } D_q f(x) = \frac{f(x) - f(qx)}{x(1-q)}.$$

▶ Lax Pair

- ▶ Degree iteration: $xP_n(x) = P_{n+1}(x) + a_n P_{n-1}(x) + b_n P_n(x).$

- ▶ q -lattice iteration:

$$P_n(qx) = P_n(x) + x(q^n - 1)P_{n-1}(x) + (c_n P_{n-2}(x) + \dots).$$

▶ Discrete Painlevé Equations

- ▶ $a_n(a_{n+1} + q^{1-n}a_n + q^2a_{n-1} + q^{3-2n}a_{n+1}a_n a_{n-1}) = (1 - q^n)q^{n-1}.$

▶ Riemann Hilbert Problem

- ▶ Analytic function with jump $Y_n^+ = Y_n^- J.$

- ▶ Transformations lead to asymptotic information about $Y_n.$

▶ Random matrices

- ▶ Classical Hermite: Eugene Wigner, Gaussian Orthogonal Ensemble.

- ▶ q -Hahn: Hexagonal tiling and gap probabilities.

References

- [1] L. Boelen, C. Smet, and W. Van Assche, *q-Discrete Painlevé equations for recurrence coefficients of modified q-Freud orthogonal polynomials*, Journal of Difference Equations and Applications **16** (2010), no. 1, pp. 37-53.
- [2] P.A. Deift, T. Kriecherbauer, K. McLaughlin, S. Venakides, and X. Zhou, *Strong asymptotics of orthogonal polynomials with respect to exponential weights*, Communications on Pure and Applied Mathematics: A Journal Issued by the Courant Institute of Mathematical Sciences **52** (1999), no. 12, 1491–1552.
- [3] A. Knizel, *Moduli spaces of q-connections and gap probabilities*, International Mathematics Research Notices **2016** (2016), no. 22, 6921–6954.
- [4] N. Joshi and T. Lasic Latimer, *On a class of q-orthogonal polynomials and the q-Riemann Hilbert Problem*, arXiv:2106.01042.
- [5] T. Lasic Latimer, *Unique positive solutions to q-discrete equations associated with orthogonal polynomials*, Journal of Difference Equations and Applications **27** (2021), no. 5, 763-775.