Probabilistic conformal blocks and their properties

Promit Ghosal

MSRI Postdoc Seminar

Joint work with G. Remy (Columbia), X. Sun (UPenn) and Y. Sun (UChicago)

October 1st, 2021

Outline

- Background on Liouville theory
- Construction of probabilistic block
- Main Result
- Proof ideas

Liouville theory and its background

QUANTUM GEOMETRY OF BOSONIC STRINGS

A.M. POLYAKOV
L.D. Landau Institute for Theoretical Physics, Moscow, USSR

Received 26 May 1981

We develop a formalism for computing sums over random surfaces which arise in all problems containing gauge invariance (like QCD, three-dimensional Ising model etc.). These sums are reduced to the exactly solvable quantum theory of the two-dimensional Liouville lagrangian. At $D = 26$ the string dynamics is that of harmonic oscillators as was predicted earlier by dual theorists, otherwise it is described by the nonlinear integrable theory.

There are methods and formulae in science, which serve as master-keys to many apparently different problems. The resources of such things have to be refilled from time to time. In my opinion at the present time we have to develop an art of handling sums over random surfaces. These sums replace the old-fashioned

representation.

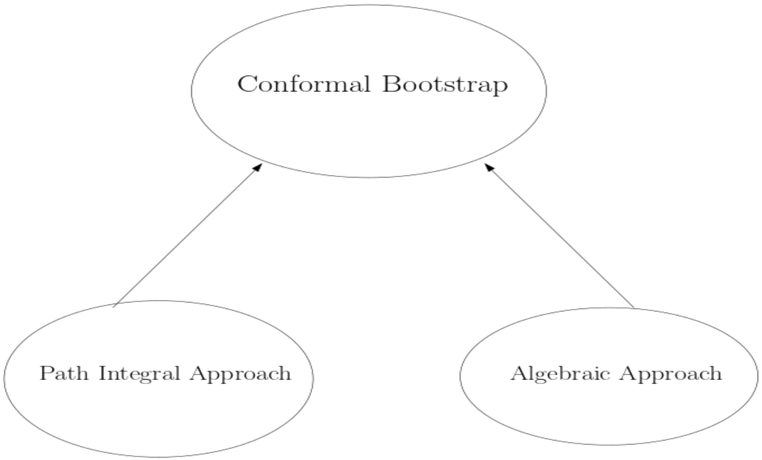
All these considerations had one essential flaw: it was not known what was exactly meant by the "free string". It has been clear, that just as the amplitudes of free particles are defined as

Let ϕ be a Liouville field. **Quantization** of ϕ is given by

$$\langle F(\phi)\rangle = \int_{\phi:\mathcal{S}\mapsto\mathbb{R}} D\phi e^{-S_L(\phi)} F(\phi),$$

where $S_L(\phi) := \int_{\mathcal{S}} (|\partial_z \phi|^2 + e^{\gamma\phi(z)})dz$ is the energy functional (Liouville action) with fundamental parameter: $\gamma \in (0,2)$.

Bootstrap framework of Liouville CFT



Belavin, Polyakov, Zamolodchikov' 84 introduced the conformal bootsrap program to combine *Polyakov's path integral approach* and the *representation theoretic* approach towards CFT.

Probabilistic framework for Liouville CFT

1. David-Kupiainen-Rhodes-Vargas '16

   Rigorously revived path integral approach of Liouville CFT on sphere and started the Bootstrap program.

   Two main components of their program are

   - Gaussian free field (GFF)
   - Gaussian multiplicative chaos (GMC)

     = Random measure, formally exp(GFF).

2. Kupiainen, Rhodes, Vargas '17 proved the DOZZ formula for fundamental (structural) constants of Liouville CFT. DOZZ is named after Dorn, Otto, Zamolodchikov and Zamolodchikov who originally proposed this formula.

3. Kupiainen, Guillarmou, Rhodes, Vargas '20 proved the conformal bootstrap on the sphere.

Goal of our talk

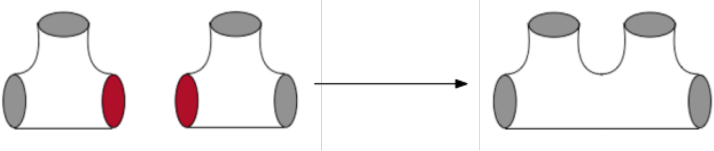
Focus on one-point function $\langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}}$ on the torus $\mathbb{T}$.

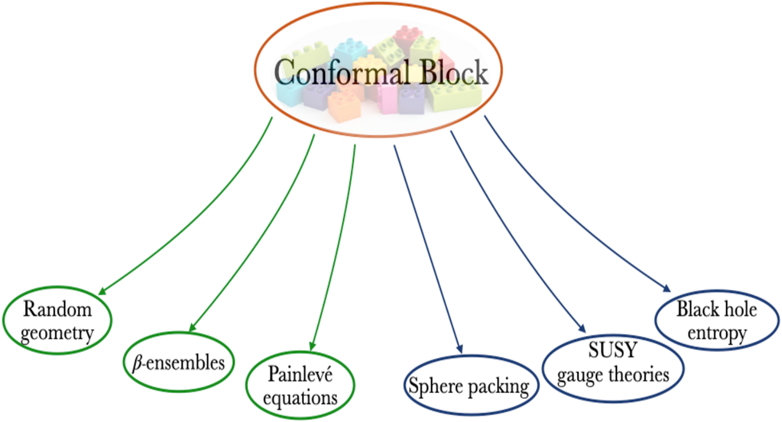
Conjectured bootstrap formula:

$$\langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}} = \int_{\mathbb{R}} dP |q|^{P^2} C_\gamma(Q - iP, \alpha, Q + iP) \mathcal{H}^{\alpha}_{\gamma,P}(q) \mathcal{H}^{\alpha}_{\gamma,P}(\overline{q})$$

- $\langle e^{\alpha\phi(0)} \rangle_{\mathbb{T}}$ defined using probability (GFF + GMC).
- $q = e^{i\pi\tau}$, $\tau \in \mathbb{H}$ is the modular parameter of $\mathbb{T}$.
- $C_\gamma(Q - iP, \alpha, Q + iP) =$ DOZZ formula, $Q = \frac{\gamma}{2} + \frac{2}{\gamma}$.
- $\mathcal{H}^{\alpha}_{\gamma,P}(q) =$ conformal block.

Goal today: understand $\mathcal{H}^{\alpha}_{\gamma,P}(q)$ and its properties using probability.




Conformal Block
Random geometry
β-ensembles
Painlevé equations
Sphere packing
SUSY gauge theories
Black hole entropy

Setup, log correlated fields

Log-correlated Gaussian field Y on $[0,1]$:

$$\mathbb{E}[Y(x)Y(y)] = -2\log|e^{2i\pi x} - e^{2i\pi y}|$$

- $Y(x)$ has an infinite variance
- Y lives in the space of distributions
- Series definition, β_n, $\tilde{\beta}_n$ i.i.d. $\mathcal{N}(0,1)$,

$$Y(x) = \sum_{n\geq 1} \sqrt{\frac{2}{n}}\Big(\beta_n \cos(2\pi n x) + \tilde{\beta}_n \sin(2\pi n x)\Big)$$

- Cut-off approximation Y_N, truncate the series at N.

Log-correlated fields with τ

Modular parameter $\tau \in \mathbb{H}$, $q = e^{i\pi\tau}$.

Log-correlated field Y_τ on $[0,1]$:

$$\mathbb{E}[Y_\tau(x)Y_\tau(y)] = -2\log|\Theta_\tau(x-y)| + 2\log|q^{1/6}\eta(q)|.$$

Decomposition $Y_\tau(x) = Y(x) + F_\tau(x)$.

For $\beta_{n,m}$, $\tilde{\beta}_{n,m}$ i.i.d. $\mathcal{N}(0,1)$,

$$F_\tau(x) = 2\sum_{n,m\geq 1} \frac{q^{nm}}{\sqrt{n}}\Big(\beta_{n,m}\cos(2\pi n x) + \tilde{\beta}_{n,m}\sin(2\pi n x)\Big).$$

$\Theta_\tau(x) = -2q^{1/4}\sin(\pi x)\prod_{k=1}^\infty(1-q^{2k})(1-2\cos(2\pi x)q^{2k} + q^{4k})$.
$\eta(q) = q^{\frac{1}{12}}\prod_{k=1}^\infty(1-q^{2k})$.

Gaussian multiplicative chaos (GMC)

For $\gamma \in (0,2)$, define on $[0,1]$ the measure $e^{\frac{\gamma}{2} Y_\tau(x)} dx$

- Cut-off approximation $e^{\frac{\gamma}{2} Y_{\tau,N}(x)} dx$
- $\mathbb{E}[e^{\frac{\gamma}{2} Y_{\tau,N}(x)}] = e^{\frac{\gamma^2}{8} \mathbb{E}[Y_{\tau,N}(x)^2]}$
- Renormalized measure: $e^{\frac{\gamma}{2} Y_{\tau,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y_{\tau,N}(x)^2]} dx$

Proposition

The following limit holds in probability, for any continuous test function f, $\forall \gamma \in (0,2)$:

$$\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} f(x) dx := \lim_{N \to +\infty} \int_0^1 e^{\frac{\gamma}{2} Y_{\tau,N}(x) - \frac{\gamma^2}{8} \mathbb{E}[Y^2_{\tau,N}(x)]} f(x) dx$$

Probabilistic conformal blocks

For $\gamma \in (0,2)$, $q \in (0,1)$, $\alpha \in (-\frac{4}{\gamma}, \frac{\gamma}{2} + \frac{2}{\gamma})$, $P \in \mathbb{R}$,

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) := \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}dx\right)^{-\frac{\alpha}{\gamma}}\right]$$

- $\gamma \in (0,2)$, link to the central charge.
- $\alpha =$ weight of marked point.
- $P =$ integration parameter of bootstrap integral.
- $\tau \in \mathbb{H}$, modular parameter of $\mathbb{T}$, $q = e^{i\pi\tau}$.

Z such that $\lim_{q\to 0}\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1$, $\lim_{P\to+\infty}\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1$.

Dotsenko-Fateev integrals for blocks

Let $-\frac{\alpha}{\gamma} = N < \frac{4}{\gamma^2}$ with $N \in \mathbb{N}$. Then

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}dx\right)^{-\frac{\alpha}{\gamma}}\right] =$$

$$C\int_{[0,1]^N}\prod_{1\leq i<j\leq N}|\Theta_\tau(x_i-x_j)|^{-\frac{\gamma^2}{4}}\prod_{i=1}^{N}\Theta_\tau(x_i)^{-\frac{\alpha\gamma}{2}}e^{\pi\gamma P x_i}\prod_{i=1}^{N}dx_i$$

First principle definition of conformal blocks

Virasoro algebra $\{L_n\}_{n\in\mathbb{Z}}$ encoding conformal symmetry:

$$L_n L_m - L_m L_n = (n-m)L_{n+m} + \tfrac{c}{12}(n-1)n(n+1)\delta_{n+m,0}\mathbf{1}.$$

Conformal blocks as a formal q-power series

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) = q^{-\frac{1}{12}}\eta(q)\,\mathrm{Tr}|_{M_{\Delta,c}}\Big(q^{-2\Delta+2L_0}\phi_{\Delta_\alpha}(1)\Big).$$

- $M_{\Delta,c}$: Verma module; $\phi_{\Delta_\alpha}(1)$: primary operator.
- $c = 1+6Q^2$, $Q = \frac{\gamma}{2}+\frac{2}{\gamma}$, $\Delta = \frac{1}{4}(Q^2+P^2)$, $\Delta_\alpha = \frac{\alpha}{2}(Q-\frac{\alpha}{2})$.

Defines a formal series, convergence not known.

Zamolodchikov's recursion

From the first principle definition, Zamolodchikov (1987) derived a recursive algorithm to compute the q-series.

Zamolodchikov's recursion

The power series in q of $\mathcal{H}^{\alpha}_{\gamma,P}(q)$ is specified by:

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1 + \sum_{n,m\geq 1} q^{2nm} \frac{R_{m,n}(\alpha)}{P^2 - P_{m,n}^2} \mathcal{H}^{\alpha}_{\gamma,P_{-n,m}}(q).$$

$$P_{m,n} = \frac{2in}{\gamma} + \frac{im\gamma}{2}, \quad R_{m,n}(\alpha) = \frac{2\prod_{k=-m}^{m-1}\prod_{l=-n}^{n-1}(Q - \frac{\alpha}{2} - \frac{k\gamma}{2} - \frac{2l}{\gamma})}{\prod_{k=-m+1}^{m}\prod_{l=-n+1}^{n}(\frac{k\gamma}{2} + \frac{2l}{\gamma})}.$$

q^2 computation: $\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1 + q^2 \frac{R_{1,1}(\alpha)}{P^2 - P_{1,1}^2} + \cdots$.

AGT correspondence

Alday, Gaiotto, Tachikawa (AGT) correspondence.

Equivalence between 2d CFT and 4d SUSY gauge theory.

Nekrasov partition function

Write $(q^{-1/12}\eta(q))^{\Delta}\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1 + \sum_{k=1}^{\infty} a_k q^{2k}$,

$$a_k = \sum_{|Y_1|+|Y_2|=k} \prod_{i,j=1}^{2} \prod_{s\in Y_i} \frac{(E_{ij}(s,P)-\alpha)(Q-E_{ij}(s,P)-\alpha)}{E_{ij}(s,P)(Q-E_{ij}(s,P))}$$

$Q = \frac{\gamma}{2} + \frac{2}{\gamma}$, (Y_1, Y_2) Young diagrams,
$E_{ij}(s,P) = iP(\delta_{i=1,j=2} - \delta_{i=2,j=1}) - \frac{\gamma}{2}H_{Y_j}(s) + \frac{2}{\gamma}(V_{Y_i}(s)+1)$.

Fateev-Litvinov '10 showed the coefficients this series obeys the Zamolodchikov's recursion.

Probability catches up with Rep. Theory

Theorem (G., Remy, Sun, Sun)

For $\gamma \in (0,2)$, $\alpha \in (0, \frac{2}{\gamma} + \frac{\gamma}{2})$, $P \in \mathbb{R}$, the q-power series for conformal block $\mathcal{H}^{\alpha}_{\gamma,P}(q)$ is convergent for $|q| < C$ for some $C > \frac{1}{2}$.

Moreover, for $q \in (0,1)$,

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}dx\right)^{-\frac{\alpha}{\gamma}}\right].$$

[Remark] The normalization Z has an explicit expression.

Applications

1. Modular Transformations: Talks about some duality between conformal blocks at τ and $-\frac{1}{\tau}$

Theorem (G., Remy, Sun, Sun), In Preparation

For $\gamma \in (0,2)$, $\alpha \in (0, \frac{2}{\gamma} + \frac{\gamma}{2})$,

$$q^{\frac{P^2}{2} - \frac{c}{24}} \mathcal{H}^q_{\gamma,P}(\alpha) = \int_{\mathbb{R}} \tilde{q}^{\frac{(P')^2}{2} - \frac{c}{24}} \mathcal{M}_\alpha(P,P') \mathcal{H}^{\tilde{q}}_{\gamma,P'}(\alpha) dP'$$

for a certain explicit **modular kernel** $\mathcal{M}_\alpha(P,P')$, where $q = e^{i\pi\tau}$ and $\tilde{q} = e^{-\frac{i\pi}{\tau}}$.

This identity for the Nekrasov's partition function is related to some quantitative version of celebrated S-duality.

2. Relation between $\lim_{\gamma\to\infty} \gamma^2 \log \mathcal{H}^{\alpha/\gamma}_{\gamma,P/\gamma}(q)$ and Painlevé tau function (Work in progress with H. Desiraju and A. Prokhorov).

Proof strategy of the main result

Tools of CFT:

- BPZ differential equations.
- Operator product expansion (OPE).

Steps of the proof:

- BPZ equations + OPE imply a system of shift equations for GMC conformal block.
- The q-series defined by Zamolodchikov's recursion obeys the same system of shift equations.
- The system has a unique solution.

BPZ equations & OPE

- CFT $\Rightarrow$ Correlation functions / conformal blocks can obey BPZ differential equations.
- Requirement: "degenerate weight" $-\frac{\gamma}{2}$ or $-\frac{2}{\gamma}$.
- Study solution space of BPZ equations
  $\Rightarrow$ non-trivial relations on GMC.
- OPE $\Rightarrow$ boundary conditions to constrain the solution space.

Summary: BPZ & OPE provides integrability of GMC.

u-deformed conformal blocks

Introduce the observable that will satisfy BPZ equation.

- Let $\chi = \frac{\gamma}{2}$ or $\frac{2}{\gamma}$.
- Let $u \in \mathbb{C}$ with $0 < \mathrm{Im}(u) < \frac{3}{4}\mathrm{Im}(\tau)$.

u-deformed conformal block

$$\psi_\chi^\alpha(u,q) := q^{\Delta_1(\chi)} \Theta_\tau'(0)^{\Delta_2(\chi)} \Theta_\tau(u)^{-l_\chi} e^{\chi P u \pi}$$
$$\times \mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2} Y_\tau(x)} \Theta_\tau(x)^{-\frac{\alpha\gamma}{2}} \Theta_\tau(u+x)^{\frac{\gamma\chi}{2}} e^{\pi \gamma P x} dx \right)^{-\frac{\alpha}{\gamma} + \frac{\chi}{\gamma}} \right]$$

for $l_\chi = -\frac{\alpha\chi}{2} + \frac{\chi^2}{2}$ and some exponents $\Delta_1(\chi)$, $\Delta_2(\chi)$.

BPZ equations and OPE for $\psi^\alpha_\chi(u,q)$

$\psi^\alpha_\chi(u,q)$ obeys the BPZ equation

$$\left(\partial_{uu} - l_\chi(l_\chi+1)\wp(u) + 2i\pi\chi^2\partial_\tau\right)\psi^\alpha_\chi(u,q) = 0.$$

$\wp(u) =$ Weierstrass's elliptic function, $l_\chi = -\frac{\alpha\chi}{2} + \frac{\chi^2}{2}$.

- The above equation is called *non-stationary* Lamé's equation (satisfied by Baxter's Q operator in eight vertex model).
- This is also related to quantum Painlevé VI via a simple change of variable.

OPE: Expansion in $u \to 0$ of $\psi^\alpha_\chi(u,q)$

$$\psi^\alpha_\chi(u,q) = \mathcal{C}_1 u^{-l_\chi}\mathcal{H}^{\alpha-\chi}_{\gamma,P}(q) + \mathcal{C}_2 u^{1+l_\chi}\mathcal{H}^{\alpha+\chi}_{\gamma,P}(q) + o(|u|^{1+l_\chi})$$

for explicit prefactors $\mathcal{C}_1$ and $\mathcal{C}_2$ depending on γ, α, P.

BPZ equations and OPE for $\psi_\chi^\alpha(u,q)$

$\psi_\chi^\alpha(u,q)$ obeys the BPZ equation

$$\left(\partial_{uu} - l_\chi(l_\chi+1)\wp(u) + 2i\pi\chi^2\partial_\tau\right)\psi_\chi^\alpha(u,q) = 0.$$

$\wp(u) =$ Weierstrass's elliptic function, $l_\chi = -\frac{\alpha\chi}{2} + \frac{\chi^2}{2}$.

- The above equation is called *non-stationary* Lamé's equation (satisfied by Baxter's Q operator in eight vertex model).
- This is also related to quantum Painlevé VI via a simple change of variable.

OPE: Expansion in $u \to 0$ of $\psi_\chi^\alpha(u,q)$

$$\psi_\chi^\alpha(u,q) = \mathcal{C}_1 u^{-l_\chi}\mathcal{H}_{\gamma,P}^{\alpha-\chi}(q) + \mathcal{C}_2 u^{1+l_\chi}\mathcal{H}_{\gamma,P}^{\alpha+\chi}(q) + o(|u|^{1+l_\chi})$$

for explicit prefactors $\mathcal{C}_1$ and $\mathcal{C}_2$ depending on γ, α, P.

From BPZ to hypergeometric equations

q-expansion + change of variable:

- $\psi^{\alpha}_{\chi}(u,q) = \sin(\pi u)^{-l_\chi} q^{\Delta(\chi)} \sum_{n=0}^{\infty} \phi^{\alpha}_{\chi,n}(w) q^n$
- $w = \sin^2(\pi u)$.

System of inhomogenous hypergeometric equations for $\phi^{\alpha}_{\chi,n}$:

$$\begin{aligned}(w(1-w)\partial_{ww} + (C - (1 + A_n + B_n)w)\partial_w - A_n B_n)\phi^{\alpha}_{\chi,n}(w) \\ = \frac{l_\chi(l_\chi+1)}{4\pi^2} \sum_{l=1}^{n} \wp_l(u) \phi^{\alpha}_{\chi,n-l}(w).\end{aligned}$$

OPE $\Rightarrow$ boundary conditions for the solution space.

$A_n = -\frac{l_\chi}{2} + i\frac{\chi}{2}\sqrt{P^2+2n}$, $B_n = -\frac{l_\chi}{2} - i\frac{\chi}{2}\sqrt{P^2+2n}$, $C = \frac{1}{2} - l_\chi$.

System of shift equations for GMC block

Write $\mathcal{H}^{\alpha}_{\gamma,P}(q) = 1 + \sum_{n=1}^{+\infty} a_n(\alpha) q^n$.

BPZ equations + OPE + q-series expansion implies

$$a_n(\alpha + \frac{\gamma}{2}) = c_n a_n(\alpha - \frac{\gamma}{2}) + G_n((a_k(\alpha))_{k\in[0,n-1]}) \quad (1)$$

$$a_n(\alpha + \frac{2}{\gamma}) = \tilde{c}_n a_n(\alpha - \frac{2}{\gamma}) + \tilde{G}_n((a_k(\alpha))_{k\in[0,n-1]}) \quad (2)$$

where G_n, $\tilde{G}_n$ are explicit linear functions.

Recursively, the system (1) + (2) has a unique solution.

(provided that $\gamma^2 \notin \mathbb{Q}$)

End of proof

Need to show that the q-series defined by Zamolodchikov's recursion also satisfies the system (1)+(2).

- **Step 1:** For $-\frac{\alpha}{\gamma} = N \in \mathbb{N}$, $N < \frac{4}{\gamma^2}$,

  $\Rightarrow$ GMC block $=$ N-fold integral involving Θ_τ.

  $\Rightarrow$ q-series defined Zamolodchikov's recursion $=$ N-fold integral via some integral trick(†).

- **Step 2:** The q-series defined by Zamolodchikov's recursion

  satisfies shift equation (1) by using (†) and analycity in γ,

  satisfies shift equation (2) by the symmetry $\frac{\gamma}{2} \leftrightarrow \frac{2}{\gamma}$.

Outlook and perspectives

Summary:

- Probabilistic construction of 1-point torus Liouville conformal block.
- Matches with the previous definitions and solve the convergence problem.
- Explore its analytic properties to prove other important conjectures.

Future directions:

- Conformal blocks in other geometry.
- Analogue of modular transformation and Nekrasov-Shatasvili quantization relation in other geometry.
- Sewing of Liouville conformal blocks.

Explicit expression for normalization Z

$$\mathcal{H}^{\alpha}_{\gamma,P}(q) = \frac{1}{Z}\mathbb{E}\left[\left(\int_0^1 e^{\frac{\gamma}{2}Y_\tau(x)}\Theta_\tau(x)^{-\frac{\alpha\gamma}{2}}e^{\gamma\pi P x}dx\right)^{-\frac{\alpha}{\gamma}}\right].$$

$$\begin{aligned} Z =& q^{\frac{1}{12}(\frac{\alpha\gamma}{2}+\frac{\alpha^2}{2}-1)}\eta(q)^{\alpha^2+1-\frac{\alpha\gamma}{2}}e^{\frac{i\pi\alpha^2}{2}}\left(\frac{\gamma}{2}\right)^{\frac{\gamma\alpha}{4}}e^{-\frac{\pi\alpha P}{2}}\Gamma(1-\frac{\gamma^2}{4})^{\frac{\alpha}{\gamma}} \\ &\times\frac{\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma}+\frac{\alpha}{2})\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}-iP)\Gamma_{\frac{\gamma}{2}}(Q-\frac{\alpha}{2}+iP)}{\Gamma_{\frac{\gamma}{2}}(\frac{2}{\gamma})\Gamma_{\frac{\gamma}{2}}(Q-iP)\Gamma_{\frac{\gamma}{2}}(Q+iP)\Gamma_{\frac{\gamma}{2}}(Q-\alpha)}. \end{aligned}$$

$$\log\Gamma_{\frac{\gamma}{2}}(z) = \int_0^\infty \frac{dt}{t}\left[\frac{e^{-zt}-e^{-\frac{Qt}{2}}}{(1-e^{-\frac{\gamma t}{2}})(1-e^{-\frac{2t}{\gamma}})} - \frac{(\frac{Q}{2}-z)^2}{2}e^{-t} + \frac{z-\frac{Q}{2}}{t}\right].$$

Integral form of $\mathcal{M}_\alpha(P, P')$

Ponsot and Teschner' 01 predicted precise form of the **modular kernel** $\mathcal{M}_\alpha(P, P')$.

$$\mathcal{M}_\alpha(P,P') = \frac{2^{3/2}}{\mathbf{i}} \frac{\sin(\mathbf{i}\pi\gamma P'/2)\sin(2\mathbf{i}\pi P'/\gamma)}{S_{\gamma/2}(\alpha/2)} \times \int_{\mathcal{C}} d\xi \frac{S_{\gamma/2}(\mathbf{i}P'/2+\alpha/2+\xi)}{S_{\gamma/2}(\mathbf{i}P'/2+Q-\alpha/2+\xi)} \times \frac{S_{\gamma/2}(\mathbf{i}P'/2+\alpha/2-\xi)}{S_{\gamma/2}(\mathbf{i}P'/2+Q-\alpha/2-\xi)} e^{-2\pi P\xi}$$