Summer Graduate Workshop - MSRI

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1 Toro's problems

Problem 1 A Radon measure μ on $\mathbb{R}^n$ is said to be doubling, if there exists a constant $C = C(n)$ depending only on n, such that for every $r > 0$ and every $x \in \mathbb{R}^n$

$$\mu(B(x, 2r)) \leq C\mu(B(x, r)).$$

Show that for any open set $U \subset \mathbb{R}^n$, and ¿0, there exists a countable collection $\mathcal{G}$ of disjoint closed balls in U such that $\operatorname{diam} \mathrm{B} \leq \delta$ for all $B \in \mathcal{G}$, and

$$\mu(U \setminus \bigcup_{B \in \mathcal{G}} B) = 0.$$

Problem 2.
Definition: Let $S \subset \mathbb{R}^n$, $m \leq n-1$, and $\epsilon \in (0, \frac{1}{4})$. Assume that $0 \in S$. We say that S has the weak ϵ- approximation property in $B_1(0)$ if $\forall \rho \in (0,1]$ and for each $Q \in S \cap B_1(0)$ there exists an m plane $L(\rho, Q)$ containing Q and such that

$$S \cap B_\rho(Q) \subset (\epsilon\rho) - \text{neighborhood of } L(\rho, Q) \cap B_\rho(Q).$$

Prove that there is a function $\beta : (0, \infty) \to (0, \infty)$ with $\lim_{t \to 0} \beta(t) = 0$ such that if S satisfies the weak ϵ- approximation property in $B_1(0)$ then

$$\mathcal{H}^{m+\beta(\epsilon)}(S \cap B_1(0)) = 0.$$

Here $\mathcal{H}^s$ denotes the s dimensional Hausdorff measure.

Problem 3. Let μ be a Borel measure on $\mathbb{R}^n$, and let $E \subset \mathbb{R}^n$ be a μ-measurable set with $0 < \mu(E) < \infty$. Show that for $s > 0$

- if
$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{r^s} < c < \infty \quad \forall x \in E,$$
then $\mathcal{H}^s(E) > 0$,
- if
$$\limsup_{r\to 0} \frac{\mu(B(x,r)\cap E)}{r^s} > c > 0 \quad \forall x \in E,$$
then $\mathcal{H}^s(E) < \infty$.

Problem 4. Let μ be a Radon measure on $\mathbb{R}^n$. Prove that $\mu << \mathcal{H}^s$ if and only if $\theta^{*,s}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}^n$.

Problem 5. Let $E \subset \mathbb{R}^n$ satisfy $0 < \mathcal{H}^s(E) < \infty$, for $0 < s < 1$. Show that the density
$$\theta^s(E,x) = \lim_{r\to 0} \frac{\mathcal{H}^s(E\cap B(x,r))}{\omega_s r^s}$$
fails to exit at almost every point of E (i.e. $\theta^s(E,x)$ exists at most in a subset of E of $\mathcal{H}^s$ measure 0).

Remark: Marstrand proved this result in 1954. Later on he showed that if $s > 0$, and $\theta^s(E,x)$ exists on a subset $F \subset E$ with $\mathcal{H}^s(F) > 0$, then s must be an integer.

Problem 6. Let μ_j, μ be Radon measures on a metric space X. Assume that for each $x \in X$, and each $j = 1, 2, ...$
$$\theta(\mu_j, x, r) = \frac{\mu_j(B_r(x))}{\omega_n r^n}, \text{ and } \theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n},$$
are non-decreasing functions of r. Assume also that μ_j converges weakly to μ, and that $x_j \to x$ as $j \to \infty$. Prove that
$$\limsup_{j\to\infty} \theta(\mu_j, x_j) \le \theta(\mu, x).$$
Here $\theta(\mu_j, x) = \lim_{r\to 0} \theta(\mu_j, x, r)$, and $\theta(\mu, x) = \lim_{r\to 0} \theta(\mu, x, r)$.

Remark: Note that in particular if $\mu_j = \mu$ for each j and $\theta(\mu, x, r)$ is a non-decreasing function of r, then the result above proves the upper semi-continuity of the density.

Problem 7. Let $M \subset \mathbb{R}^m$, $0 < n < m$, and $\mu = \mathcal{H}^n \llcorner M$. Assume that μ is a Radon measure, and that for each $x \in M$ $\theta(\mu, x, r) = \frac{\mu(B_r(x))}{\omega_n r^n}$ is a non-decreasing function of r. Let $\lambda_j > 0$ be a sequence converging to 0 as $j \to \infty$. For $x \in M$, let

$$M_j = \frac{1}{\lambda_j}(M - x) = \{y = \frac{1}{\lambda_j}(z - x) : z \in M\},$$

and

$$\mu_j = \mathcal{H}^n \llcorner (M_j \cap B_1(0)).$$

Show that for each j, μ_j is a Radon measure. Prove that there exists a subsequence μ_{j_k} of μ_j that converges weakly to a Radon measure ν, and that

$$(*) \qquad \theta(\mu, x) = \theta(\nu, 0).$$

Note that in particular $(*)$ asserts that $\lim_{r\to 0} \theta(\nu, 0, r)$ exits.

Remark: The situation described in Problem 3 occurs when M is a minimal n-dimensional submanifold of $\mathbb{R}^m$. In that case $\nu = \mathcal{H}^n \llcorner C$, where C is a cone of vertex 0. C is a tangent cone of M at x. As defined this cone depends on the subsequence λ_{j_k}. One of the big open questions in the subject is whether there is a unique tangent cone. Moreover the set $\{x \in M : \theta(\mu, x) = 1\}$ is open and smooth. The set $\{x \in M : \theta(\mu, x) > 1\}$ is a closed set of Hausdorff dimension at most $n - 1$.

Problem 8.
Definition: Let μ be a Radon measure in $\mathbb{R}^n$. Set, for $x \in \mathbb{R}^n$,

$$M_\mu f(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f| \, d\mu,$$

if f is a μ-measurable function, and

$$M_\mu \nu(x) = \sup_{r>0} \frac{\nu(B(x,r))}{\mu(B(x,r))},$$

if ν is a Radon measure in $\mathbb{R}^n$.

- Show that there exists a constant $C < \infty$ depending only on n, with the following property: if μ and ν are Radon measures in $\mathbb{R}^n$, then

$$\mu\left(\{x \in \mathbb{R}^n : M_\mu \nu(x) > t\}\right) \le C t^{-1} \nu(\mathbb{R}^n).$$

- Show that for $1 < p < \infty$ there exists a constant $C_p < \infty$, depending only on n and p with the following property: if μ is a Radon measure in $\mathbb{R}^n$, then

$$\int (M_\mu f)^p \, d\mu \le C_p \int |f|^p \, d\mu,$$

  for all μ-measurable functions f.

Problem 9. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a Lipschitz map, and $A \subset \mathbb{R}^n$ be an $\mathcal{H}^n$-measurable set. Show that $\Theta^n_*(f(A), x) > 0$ for $\mathcal{H}^n$ almost every $x \in f(A)$.

Problem 10.
Definition 1: A map $f : A \to B$, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ is said to be bi-Lipschitz if f is Lipschitz and it has a Lipschitz inverse $f^{-1} : B \to A$.
Definition 2: A set $E \subset \mathbb{R}^n$ is said to be an Ahlfors s-regular set for some $0 < s \leq n$, if there exists a constant $C > 1$ so that for every $r > 0$ and each $x \in E$,

$$C^{-1} r^s \leq \mathcal{H}^s(E \cap B(x, r)) \leq C r^s.$$

Show that the image of an Ahlfors s-regular set by a bi-Lipschitz map is an Ahlfors s-regular set.

Problem 11. Let $S \subset \mathbb{R}^n$, $m \leq n - 1$, and $\epsilon \in (0, \frac{1}{2})$. Let $0 \in S$. Assume that there exists an m plane L containing the origin, such that $\forall \rho \in (0, 1]$ and for each $x \in S \cap B(0, 1)$

$$S \cap B(x, \rho) \subset (\epsilon\rho) - \text{neighborhood of } (L + x) \cap B(x, \rho).$$

Prove that $S \cap B(0, \frac{1}{4})$ is contained in a Lipschitz graph. Give an estimate for the Lipschitz constant of the corresponding function.

Problem 12. Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be Lipschitz, $n \geq m$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be an $\mathcal{H}^n$-summable function. Assume that $\sup_{x \in \mathbb{R}^n} |f(x)| \leq R$, and that $g \geq 0$. Show that for each $\mathcal{H}^n$-measurable set $A \subset \mathbb{R}^n$, there exists a set $S \subset B(0, R) \subset \mathbb{R}^m$ ($S = S(g, f, A)$), such that $\mathcal{H}^m(S) \geq \frac{1}{2}\mathcal{H}^m(B(0, R))$, and for each $y \in S$

$$\int_{f^{-1}(y) \cap A} g \, d\mathcal{H}^{n-m} \leq \frac{2}{\mathcal{H}^m(B(0, R))} \int_A g \, Jf \, d\mathcal{H}^n.$$

Problem 13.. Let $U \subset \mathbb{R}^n$ be an open set, let $u \in BV(U)$ and $f \in C^\infty_C(U)$. Then $fu \in BV(U)$ and $\forall \varphi \in C^1_c(U, \mathbb{R}^n)$,

$$\int_U \varphi \, d[D(fu)] = \int_U \varphi f \, d[Du] + \int_U u\varphi \cdot Df \, dx,$$

i.e. $D(fu) = uDf + fDu$ in the distribution sense. Here if $u \in BV(U)$, $d[Du] = \sigma d\|Du\|$, where $\|Du\|$ is the variation measure of u, and σ is the $\|Du\|$-measurable function that appears in the structure theorem for BV functions.

Problem 14. Let N be a C^1 n-submanifold in $\mathbb{R}^{n+k}$. Let $\theta : N \to \mathbb{R}$ be an $\mathcal{H}^n$ measurable function. Let $\eta_{x,r}N = \frac{1}{r}(M - x)$. Prove for $\mathcal{H}^n - a.e.$ $x \in N$ and all $f \in C_c(\mathbb{R}^{n+k})$

$$\lim_{r\to 0} \int_{\eta_{x,r}N} f(y)\theta(ry + x)\, d\mathcal{H}^n(y) = \theta(x) \int_{T_xN} f(y)\, d\mathcal{H}^n(y).$$

Here T_xN denotes the tangent plane to N at x.

Problem 15. Let μ be a Radon measure on $\mathbb{R}^n$. Assume that for $a \in \operatorname{support} \mu = \Sigma$

$$1 \le \limsup \frac{\mu(B(a, 2r))}{\mu(B(a, r))} < \infty. \tag{1}$$

1. Show that for $\tau \ge 1$ and $a \in \Sigma$

$$1 \le \limsup \frac{\mu(B(a, \tau r))}{\mu(B(a, r))} < \infty.$$

2. Prove that if there exit $\kappa > 1$ and $R > 0$ such that for $r \in (0, R)$ and all $a \in \Sigma$

$$\frac{\mu(B(a, 2r))}{\mu(B(a, r))} \le \kappa \tag{2}$$

then for any measure ν obtained as a weak limit of a sequence

$$(\mu(B(a, r_i)))^{-1} T_{a,r_i\#}\mu \text{ where } T_{a,r_i\#}\mu(E) = \mu(r_iE + a) \text{ for } E \subset \mathbb{R}^n \text{ Borel}$$

the following statement holds: $x \in \operatorname{support} \nu$ if and only if there exists a sequence $x_i \in T_{a,r_i}(\Sigma)$ such that $x_i \to x$.

2 DeLellis's problems

Problem 1. $U \subset \mathbb{R}^n$ is a convex open set.

$$W^{1,\infty}(U) = \{u \in L^\infty_{loc} : Du \in L^\infty\};$$

$$\operatorname{Lip}(U) = \{u \in C(U) : \exists L \text{ with } |u(x) - u(y)| \le L|x - y| \forall x, y \in U\}.$$

Show that $W^{1,\infty}(U) = \operatorname{Lip}(U)$.

Problem 2. Let $U \subset \mathbb{R}^n$ be open and $u \in W^{1,p}(U)$, with $p > n$. Prove that u is differentialbe a.e.. Show a map $u \in W^{1,n}(U)$ which is not differentiable a.e..

Problem 3. Prove the Cauchy-Binet formula: if $m \geq n$ and M is an $m \times n$ matrix, then

$$\det(L^t \cdot L) = \sum_{n \times n \text{ submatrices } M \text{ of } L} (\det M)^2 .$$

Problem 4. Prove the area and coarea formulas for linear maps.

Problem 5. Prove that $BV(U)$ is a Banach space.

Problem 6. Prove that for every $u \in BV(U)$ there exists a sequence $\{u_k\} \subset BV(U) \cap C^\infty(U)$ such that $u_k \to u$ strongly in L^1 and $\|Du_k\|(U) \to \|Du\|(U)$.

Problem 7. Let $U = \{x \in \mathbb{R}^n : x_n > 0\}$. For $f \in BV(U)$ define

$$\frac{1}{\varepsilon} \int_0^\varepsilon f(x', x_n)\, dx_n .$$

Prove that $\{f_\varepsilon\}$ is Cauchy in L^1.

Problem 8. Let $f \in BV(U)$. Prove

$$\|Df\|(A) = \int_{-\infty}^{\infty} \|\partial\{f > t\}\|(A)\, dt$$

for every Borel set $A \subset U$.

Problem 9. $I \subset \mathbb{R}$ interval, (E, d) separable metric space. Define $BV(I, E)$ following Ambrosio (see lecture). Define $TV(I, E)$ as the set of measurable functions $u : I \to E$ such that

$$TV(u) := \sup_{N \in \mathbb{N}, x_0 < x_1 < \ldots < x_N \in I} \sum_{i=1}^{N} d(u(x_i, u(x_{i-1}) < \infty .$$

Prove that $BV(I, E) = TV(I, E)$ (i.e. that every $u \in TV(I, E)$ belongs to $BV(I, E)$ and for every $u \in BV(I, E)$ there is $\tilde{u} \in TV(I, E)$ such that $\tilde{u} = u$ a.e.).

Problem 10 When $E = \mathbb{R}$ prove that $\|Du\|(I) = TV(\tilde{u})$ where $\tilde{u}$ is the precise representative (see lecture).

Problem 11. Assume $\{\mu_i\}_{i\in I}$ is a (not necessarily countable!) collection of nonnegative measures on a Borel set $E \subset \mathbb{R}^n$ with the property that there is a measure μ with $\mu_i \leq \mu$ $\forall i \in I$. For every Borel set $F \subset E$ define

$$\nu(F) = \sup \left\{ \sum_{n=0}^{\infty} \mu_{i_n}(F_n) : \{F_n\} \text{ is a Borel partition of } F \ , \{i_n\} \subset I \right\} .$$

Show that ν is a measure. Show that ν is the smallest measure with the property that $\mu_i \leq \nu$ $\forall i \in I$.

Problem 12. Let $C_\alpha \subset \mathbb{R}^2$ be the cone

$$\{(x_1, x_2) : |x_2| \geq \alpha |x_1|\} .$$

Prove the existence of a Borel set $K \subset \mathbb{R}^2$ such that

- $0 < \mathcal{H}^1(K) < \infty$;
- $$\lim_{r\downarrow 0} \frac{\mathcal{H}^1(K \cap B_r(x) \cap (C_\alpha + x))}{r} = 0 \qquad \text{for all } \alpha \text{ and } \mathcal{H}^1\text{–a.e. } x.$$
- K is not rectifiable.

Hint: look at graphs of suitable functions.