MSRI Graduate Summer School Lecture 1: Intro to Cluster Algebra

Reference: Cluster Algebras I, II, III, IV by Fomin + Zelevinsky

Cluster algebras: class of commutative rings, introduced in 2000 by Fomin & Zelevinsky

Original motivation: Lusztig's dual canonical basis & total positivity

Since 2000, cluster algebra structures have been connected to:
- coordinate rings of Grassmannians, flag varieties, other homogeneous spaces...
- quiver reps
- Teichmüller theory
- invariant theory
- tropical calculus
- Poisson geometry
- Lie theory
- combinatorics (associahedra, perfect matchings, cluster complex...)
- integrable systems
- total positivity

Cluster Algebra Portal:

www.math.lsu.edu/~fomin/cluster.html

Reference for conferences, software, etc,

Fall 2012: Semester-long program on cluster algebras
Def: (F + Z) A cluster algebra $A$ is a certain subalgebra of $K(x_1, ..., x_n)$, the field of rational functions over $\{x_1, ..., x_n\}$. Generators are constructed by a series of exchange relations which in turn induce all relations satisfied by the generators.

Def: An $n \times n$ integral matrix $B$ is skew-symmetrizable if there exist $d_1, d_2, ..., d_n \in \mathbb{Z}^+$ such that $d_i b_{ij} = -d_j b_{ij} \quad \forall i, j$

Any skew-symmetric matrix is skew-symmetrizable. More generally, one can start from skew-symmetric matrix $B$ and scale the columns by pos. integers $d_1, ..., d_n$.

We can associate a (coefficient-free) cluster algebra $A(B)$ to any such matrix $B$.

Start w/ a seed $(\{x_1, ..., x_n\}, B)$.

From this seed we can mutate in each of $n$ directions, obtaining $n$ more seeds.

Columns of $B$ encode the exchange relations:

For $k \in \{1, ..., n\}$, $X_k X_k' = \prod_{b_{ik} > 0} X_i^{b_{ik}} + \prod_{b_{ik} < 0} X_i^{1/b_{ik}}$

This defines a new cluster variable $X_k'$.

For $k \in \{1, ..., n\}$, if another seed for $A$ consists of the clusters $\{x_{k_1}, ..., x_{k_r}, x_k\}$ and matrix $M_k(B)$, where
\[ M_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j \\ b_{ij} & \text{if } b_{ik} b_{kj} \leq 0 \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0 \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0 \end{cases} \]

(M_k(B) is again skew-symmetrizable)

Start from the initial seed \( \mathbf{b} \) and apply all possible sequences of mutations: this produces the set of all cluster variables (possibly infinite).

Def: The cluster algebra \( \mathcal{A}(B) \) is the subalgebra of \( k(x_1, \ldots, x_n) \) generated by all cluster variables.

Example 1: Rank 2 cluster algebras:
Let \( F = \mathbb{Q}(y_1, y_2) \) - rational functions in \( y_1 \) and \( y_2 \).

Fix \( b, c \in \mathbb{Z}^+ \) and define \( B = \begin{pmatrix} 0 & -b \\ c & 0 \end{pmatrix} \).

Then \( y_1 y_1' = y_2^c + 1 \). We refer to \( y_1' \) as \( y_0 \).

Also, \( y_2 y_2' = y_1^b + 1 \). Refer to \( y_2' \) as \( y_3 \).

\[ M_1(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \quad \text{and} \quad M_2(B) = \begin{pmatrix} 0 & b \\ -c & 0 \end{pmatrix} \]

\[ M_1^2 = M_2 \quad \text{and} \quad M_1 \circ M_2(B) = M_2 \circ M_1(B) = B \]

Our cluster variables are \( \{y_1^i y_2^j \mid i, j \in \mathbb{Z} \} \) - they satisfy
\[ Y_{m+1} = \begin{cases} \frac{Y_m + 1}{Y_{m-1}} & m \text{ odd} \\ \frac{Y_m^2 + 1}{Y_{m-2}} & m \text{ even} \end{cases} \]

The cl. alg. \( A(b,c) \) is the subring of \( F \) gen. by the \( y_m \).

Cluster vars: \( y_m \)

Exchange relations: (note: 3-term)

Clusters: \( \{y_m, y_{m+3} \} \)

Exchange graph: \( \cdots \bullet \{y_0, y_1\} \bullet \{y_1, y_2\} \bullet \{y_2, y_3\} \bullet \{y_3, y_4\} \cdots \)

Note: one can express every cluster variable as a rational expression in terms of the variables of a single cluster.

Laurent phenomenon (F-Z): Given any seed \( (\{x_1, \ldots, x_n\}, B) \) for any cluster algebra and any cluster variable \( x \), one can express \( x \) as a Laurent poly in variables \( \{x_1, \ldots, x_n\} \).

Positivity conjecture (F-Z): All coefficients in the above Laurent expansion are positive.

Ex: let \( b = 1, c = 1 \). Start w/ \( \{y_1, y_2\} \), then

\[ y_3 = \frac{y_2 + 1}{y_1}, \quad y_4 = \frac{y_3 + 1}{y_2} = \frac{y_2 + 1}{y_1} + \frac{1}{y_2} = \frac{y_3 + y_2 + 1}{y_1 y_2} \]
\[ y_5 = \frac{y_4 + 1}{y_3} = \frac{y_4 + y_2 + 1}{y_3 y_2} + 1 \]
\[ = \frac{y_4 + y_2 + y_3 + 1}{y_2 (y_2 + 1)} = \frac{y_1 + 1}{y_2} \]

\[ y_6 = \frac{y_5 + 1}{y_4} = \frac{y_5 + y_2 + 1}{(y_3 + y_2 + 1) y_2} = y_1 ! \] Periodic.

\[ \text{Note: all Laurent polynomials, all coeff's pos.} \]

Only finitely many cluster algebras so finite type. This is of type \( A_2 \).

**Example 2:** Consider a polygon \( P \) with \( n+3 \) sides, choose any triangulation \( \mathcal{T} \).

\( \text{(Will have } n+3 \text{ boundary segments, } n \text{ diagonals).} \)

\[ \text{Note: The set of all triangulations of an } (n+3) \text{-gon are connected by elementary moves called flips:} \]

\[ \text{flips:} \]

\[ \text{Eg there are 5 triangulations of a pentagon} \]

\[ \text{Diagram of 5 triangulations of a pentagon} \]
Can associate an $n \times n$ matrix $B(T)$ to $T$. First label the $n$ diagonals of $T$ from 1 to $n$.

$B(T) = (b_{ij})$ where

$$b_{ij} = \# \left\{ \text{triangle w/ sides } i \to j, j \text{ following } i \text{ in clockwise order} \right\}$$

$$-\# \left\{ \text{triangle w/ sides } j \to i, i \text{ following } j \text{ in counterclockwise order} \right\}$$

\[ i \quad j \]

Example:

$T =$

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  t_1  t_2  t_3
  t_3  t_1  t_2
  t_2  t_3  t_1
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$B(T) = \begin{pmatrix}
  1 & \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}
\end{pmatrix}$

This gives a construction of a cluster alg $A(B(T))$ assoc. to each triangulation $T$ of a polygon.

Let's consider $M_1(B(T))$.

From $M_k(B)_{ij} =$

$$\begin{cases} 
-\text{bi}_j & \text{if } k=i \text{ or } k=j \\
\text{bi}_j & \text{if } bi_k b_{kj} \leq 0 \\
\text{bi}_j + bi_k b_{kj} & \text{if } bi_k, b_{kj} > 0 \\
\text{bi}_j - bi_k b_{kj} & \text{if } bi_k, b_{kj} < 0 
\end{cases}$$

we get $M_1(B(T)) = \begin{pmatrix}
  0 & -1 & 1 \\
  1 & 0 & 0 \\
-1 & 0 & 0 
\end{pmatrix}$

Suppose we perform a flip in the triangulation, i.e.

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  [ ]  [ ]
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local picture
eg. let's flip $t_i$.

\[ T' = \begin{array}{c}
  t_2 \\
  t_i \\
  t_3 \\
\end{array} \quad \quad B(T') = \begin{pmatrix}
  0 & -1 & 1 \\
  1 & 0 & 0 \\
 -1 & 0 & 0 \\
\end{pmatrix} \]

Claim: Let $T$ be a triangulation and $T'$ be a new triangulation obtained by flipping $t_i$. Then $B(T') = M_i(T)$.

Con: Given a $(n+3)$-gon and a triangulation $T$, the cluster alg $A(B(T))$ does not depend on $T$ only on the number $n+3$.

Thm: We have bijections

- cluster variables $X_0 \leftrightarrow$ diagonals $\gamma$ of $(n+3)$-gon
- clusters $\leftrightarrow$ triangulations
- exchange relation $\leftrightarrow$ flips

\[
\begin{array}{c}
  \alpha \\
  \beta \\
\end{array} \quad \quad \text{flip} \quad \quad \begin{array}{c}
  \gamma \\
  \delta \\
\end{array}
\]

Exchange relation: $X_0 X_n = X_2 X_3 + X_0 X_5$.

Next week: Gregg will explain more about how to generalize this from polygons to arbitrary surfaces.
Sometimes one would like to assign a variable to the boundary segment in the polygon. Such a variable will never get flipped, so will be present in every triangulation (cluster).

To do so, use cluster algebras with frozen variables (coefficients).

To define such a cluster use rectangular matrix $B$, whose top $n \times n$ part is skew-symmetric. Start with seed $(\underbrace{x_1, x_2, \ldots, x_n, x_{n+1}, \ldots, x_{m^2}}_{\text{cluster}}, \underbrace{\text{coeff}}_{\text{coeff}})$.

We only mutate in directions $1, 2, \ldots, n$.

Columns of $B$ encode the exchange relations:

For $k \in \{1, \ldots, n\}$,

$$X_k X_k^{-1} = \prod_{b_{ik} > 0} X_i^{-1} + \prod_{b_{ik} < 0} X_i^{-1}$$

Mutating a matrix works same:

$$M_k(B)_{ij} = \begin{cases} -b_{ij} & \text{if } k = i \text{ or } k = j, \\ b_{ij} & \text{if } b_{ik} b_{kj} \leq 0, \\ b_{ij} + b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} > 0, \\ b_{ij} - b_{ik} b_{kj} & \text{if } b_{ik}, b_{kj} < 0. \end{cases}$$
Def: Let $\Sigma = (\underbrace{x_1, x_2, \ldots, x_n}_{\text{clust.}}, \underbrace{x_{n+1}, \ldots, x_m}_{\text{coeff}}, B)$ be a seed. Then the cluster algebra $A(\Sigma)$ is the $\mathbb{Z}[x_{n+1}, \ldots, x_m]$-subalgebra of $\mathbb{Z}[x_1, x_2, \ldots, x_m, (x_1, \ldots, x_n)]$ generated by all cluster variables.

(If more time, mention quiver mutation)
1. Check that if \( B = (c, -b) \) and we denote the initial cluster variables as \( (y_1, y_2) \), then the set of all cluster variables is in bijection with \( \mathbb{Z} \) (denote them by \( y_m, m \in \mathbb{Z} \)), and they satisfy:

\[
\begin{align*}
y_m \cdot y_{m+1} &= \begin{cases} 
    y_m^b + 1 & \text{if } m \text{ odd} \\
    y_m^c + 1 & \text{if } m \text{ even}
\end{cases}
\end{align*}
\]

2. For which \( b \) and \( c \in \mathbb{Z} \) is the set of cluster variables finite?

3. Prove the following claim.

Claim: Let \( T \) be a triangulation, and \( T' \) be a new triangulation obtained by flipping \( t_i \). Then \( B(T') = M_i(T) \).

4. Explicitly compute all cluster variables and seeds associated to the cluster algebra coming from a pentagon.